The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 9: Competition and hidden knowledge

9.1 Screening

9.1.1 Screening by means of a non-productive action

Solution 9.1.1.1.

(A) Suppose the firm wants to hire both agents. Then it will offer $w = 3$ per worker. The firm has an incentive to do this as long as the expected marginal product of labor per worker when both types are employed is higher than the wage:

$$(1 - p) + 4p \geq 3,$$

which solves to $p \geq 2/3$.

When $p < 2/3$, the firm will not set $w = 3$. If the firm sets $w = 2$, it will only attract types whose marginal productivity of labor is smaller than 2. Hence the firm will not hire any worker.

(B) First, with screening the low types must remain unemployed because their marginal product of labor is lower than their outside option, and the former is the maximum amount a firm is possibly to pay. Hence the firm wants to find a contract (z, w) that induces only the high types to work. Formally, the firm solves

$$\max_{(z,w)} 4 - w$$

s.t.

$$w - z \leq 2$$
$$w - \frac{z}{4} \geq 3.$$

The first inequality is the low type's IR constraint, which prevents the low type to be employed. The second inequality is the high type's IR constraint, which makes sure they are employed.

Suppose (z^*, w^*) solves the problem. Then evidently the high type's IR constraint must be binding, otherwise we can reduce w by a little, which continues to keep the low type's IR constraint satisfied and increases firm's profit. Substituting in $w = 3 + z/4$ into the maximization problem we can get

$$\max_{z} 1 - \frac{z}{4}$$

s.t.

$$3 - \frac{3}{4}z \leq 2.$$

Hence $z^* = 4/3$ and $w^* = 10/3$.

In equilibrium, the firm is making a profit of $4 - 10/3 > 0$. Both the high type and the low type gets a utility equal to their outside options.

(C) In the competitive case, the low types will still remain unemployed, but the high type can receive a wage equal to their marginal product of labor $w^* = 4$. To ensure the low type is unemployed, we need to choose z such that $4 - z \geq 2$. Any $z > 2$ is not an equilibrium since a firm can deviate to offer a z' such that $2 \leq z' < z$, which attracts the high types. Hence in equilibrium $z^* = 2$.

To compare, the competitive environment drives firms' profit to zero, but the high types now enjoy a utility of $4 - 2/4 = 7/2 > 3$, which is higher than their outside option. The low type remains the same.

(D) Denote the outside option of the high type by a. Then we solve again the maximization problem in (B) with the high type's constraint being $w - z/4 \geq a$. Then we will get

$$(z^*, w^*) = (\frac{4(a-2)}{3}, \frac{4a-2}{3}).$$

The firm has an incentive to offer such contract only when

$$4 \geq \frac{4a-2}{3},$$

which reduces to

$$a \leq \frac{7}{2}.$$

(E) No one will be better off or worse off. To see this, write $C(z, \theta) = kz/\theta$. In the monopolistic case the maximization problem is then

$$\max_{(z,w)} 4 - w$$

s.t.

$$w - kz \leq 2$$
$$w - \frac{kz}{4} \geq 3.$$

Substituting $w = 3 + kz/4$ into the problem to get

$$\max_{z} 4 - kz/4$$

s.t.

$$1 \leq \frac{3}{4}kz$$

Hence w remains the same. The optimal z^* is scaled back by $1/k$, which cancels with the k-times cost.

In the competitive case the high type receives $w = 4$. To prevent low type from employment and deviation of fimrs we need $4 - 2z = 2$, or $z = 1$. The high type now gets $4 - 1/2 = 7/2$, which is also the same as before.

Solution 9.1.1.2.

(A) An individual who spends $T(z, \theta)$ years in school can earn an income stream from time T with present discounted value $W(z)$. Then, discounting to time $t = 0$ using the discount rate r the individual's wealth is

$$U(z, W(z); \theta) = e^{-rT(z,\theta)}W(z).$$ [1]

Taking logs and noting that $T = az/\theta$, we get

$$lnU(z, W(z); \theta) = \ln W(z) - \frac{raz}{w}.$$

Assuming $W(z)$ is differentiable, a type θ worker chooses $z = Z(\theta)$ satisfying the first order condition $U'(z) = 0$. Hence

$$\frac{W'(z)}{W(z)} = \frac{ra}{\theta}. \tag{1}$$

Also, the present value of the profit on such work is $\Pi = \theta - W(z)$, and competition drives the profit to zero. Hence

$$W(z) = \theta \tag{2}$$

(B) Combining (1) and (2) to get

$$W'(z) = ra.$$

Since the lowest type, type β, has $Z(\beta) = 0$, we get $W(0) = \beta$. Using this boundary condition we can get

$$W(z) = raz + \beta. \tag{3}$$

[1] This is the expression for discounting when time is continuous.

(C) Substituting $W(Z(\theta)) = \theta$ and $T = aZ(\theta)/\theta$ into (3) to get

$$\theta = rT\theta + \beta.$$

Hence

$$W^*(t) = \theta = \frac{\beta}{1 - rt}.$$

9.1.2 Screening with productive actions

Solution 9.1.2.1.

(A) As discussed in the text, in the competitive equilibrium the equilibrium contract $(w^*(\theta), z^*(\theta))$ satisfies the tangency condition

$$-\frac{\partial U/\partial z}{\partial U/\partial w} = \frac{\partial V}{\partial z}(z; \theta).$$

Hence

$$\frac{2z}{\theta} = \frac{\theta}{2},$$

or

$$z = \frac{\theta^2}{4}.$$

(B) Simply plug in $\theta = 1, 2, 3$ into $z^*(\theta) = \theta^2/4$.

(C) First we compute the equilibrium wage $w^*(\theta)$ under full information. Since $z^*(\theta) = \theta^2/4$, zero profit condition gives

$$w^*(\theta) = \frac{\theta(1+z)}{2} = \frac{\theta}{2}\left(1 + \frac{\theta^2}{4}\right).$$

Thus, the types $\theta = (1, 2, 3)$ choose $z^* = (\frac{1}{4}, 1, \frac{9}{4})$ and get wages $w^* = (\frac{5}{8}, 2, \frac{39}{8})$, respectively. The equilibrium utility for each type is then $U_f(\theta_1) = 5/8 - 1/16 = 9/16$, $U_f(\theta_2) = 2 - 1/2 = 3/2$, $U_f(\theta_3) = 39/8 - 27/16 = 51/16$.

When θ is not observable, the lowest type will still get the same contract as the full information case in the screening equilibrium. For $\theta = 2$ type, the contract $(w(2), z(2))$ he gets will be on the firm's zero-profit curve for type 2 that makes type $\theta = 1$ indifferent between his own contract. Hence we have

$$w(2) = 1 + z(2)$$
$$w(2) - z(2)^2 = 9/16.$$

Solving to get $z = (2+\sqrt{11})/4$[2] and $w = 1+z$. The equilibrium utility for type $\theta = 2$ under screening equilibrium is then $U_s(\theta_2) = 1.44 < U_f(\theta_2) = 1.5$.

The highest type $\theta = 3$ will receive a contract on the zero-profit line for type 3 that makes type 2 indifferent. Hence

$$w(3) = \frac{3}{2}(1 + z(2))$$
$$w(3) - z(3)^2/2 = 1.44.$$

This solves to $z = 3.04$ and $w = 6.06$.

Hence each type higher than the lowest one are now choosing education-levels that are greater than productively optimal, implying that their equilibrium utilities decrease relative to the full-information one.

Solution 9.1.2.2.

(A) The tangency condition again shows that

$$1 = \frac{z}{\theta},$$

so $Z^*(\theta) = \theta$. The zero0profit condition then shows

$$w = V(z, \theta) = \theta + z = 2\theta.$$

Hence $W^*(Z(\theta)) = 2Z^*(\theta)$.

(B) To show that the full information equilibrium is also a screening equilibrium, it suffices to show that the low type is indifferent between his contract and that of the high type. The low type's contract is $(z, w) = (1, 2)$, which yields a utility of $3/2$ to the low type. The high type's contract is $(z, w) = (3, 6)$, and if the low type takes it, he gets

$$U((3, 6); 1) = 6 - \frac{3^2}{2} = \frac{3}{2}.$$

To see that $\hat{\theta} = 3$ is a borderline case, see Figure 1.

The indifference curve of type 1 agent intersects with the equilibrium wage curve $W(Z) = 2Z$ at two points: $(z, w) = (1, 2)$ and $(z, w) = (3, 6)$. For $\hat{\theta} < 3$, type 1 agent will prefer the type 2's full information contract. Hence they are not screening equilibria.

[2] We take the positive solution since education is never negative.

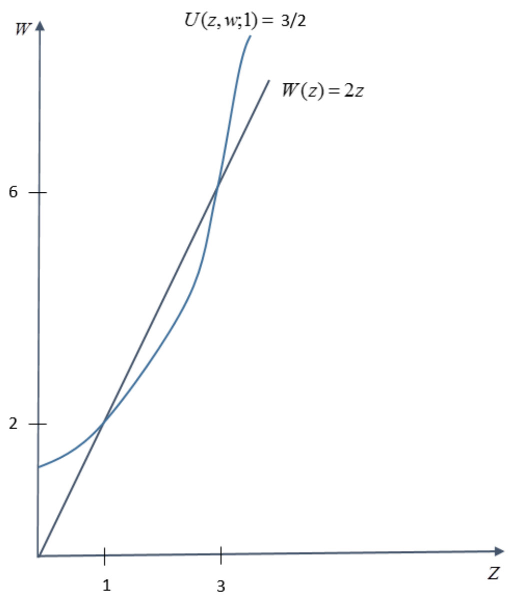

Figure 1: Ex 9.1.2.2(B)

Solution 9.1.2.3.

(A) The entrepreneur needs to raise z dollars. To do so he must offer outside investors a share in the present value of the firm so that the risk-neutral shareholders achieve r, the riskless market return. That is:

$$sf(z,\theta) = z(1+r)$$

He sells shares with a value of z and retains insider "promoter stock" with a value of z^*. The value of the firm is then:

$$P = z + z^*$$

The outsider share s is therefore z/P and so the entrepreneur's expected return is

$$U(z,P;\theta) = \left(\frac{z^*}{P}\right)\frac{f(z,\theta)}{1+r} = \left(1-\frac{z}{P}\right)\frac{f(z,\theta)}{1+r} \tag{4}$$

(B) If θ is observable, the value of the firm is $P(z,\theta) = f(z,\theta)/(1+r)$. Substituting this to (4) to get

$$U(z,P;\theta) = \frac{f(z,\theta)}{1+r} - z$$

The entrepreneur maximizes his profit, hence the optimal z^* satisfies

$$\frac{1}{1+r}\frac{\partial f}{\partial z} = 1.$$

(C) Taking logs for both sides of (4) to get

$$\ln U = \ln\left(1 - \frac{z}{P}\right) + \ln f(z,\theta) - \ln(1+r).$$

Differentiate with respect to P and z to get

$$\begin{aligned}\frac{1}{U}\frac{\partial U}{\partial P} &= \frac{z/P^2}{1 - z/P} \\ \frac{1}{U}\frac{\partial U}{\partial z} &= \frac{-1/p}{1 - z/p} + \frac{\partial f/\partial z}{f}.\end{aligned}$$

Consequently,

$$\left.\frac{dP}{dz}\right|_u = \frac{-\partial U/\partial z}{\partial U/\partial P} = \frac{P}{z} - \frac{P^2}{z^2}\left(1 - \frac{z}{P}\right)\frac{z}{f}\frac{\partial f}{\partial z}.$$

(D) By hypothesis, output elasticity is increasing in θ. Therefore

$$\frac{\partial}{\partial \theta}\left(\left.\frac{dP}{dz}\right|_U\right) < 0.$$

9.2 Reactive equilibrium

Solution 9.2.1.

(A) To show $e = (1/4, 3/4)$[3] is not a reactive equilibrium, consider a deviation of player 1 to $x = 1/2$. This is a profitable deviation, and any further deviation of player 2 will not make player 1 worse off than under e, since taking the position $x = 1/2$ guarantees at least half of the customer no matter what the other player does. Hence by definition e is not a reactive equilibrium.

(B) To show there is no NE, we will show first that there is no pooling equilibrium, where each firm chooses the same location. To see this, suppose all firms choose $x \in [0, 1]$. If $x = 1/2$, player 1 can deviate to $1/2 - \epsilon$ and earn a profit very close to 1/2 instead of 1/3. If $x \neq 1/2$ player 1 can deviate to 1/2 and earn more than 1/2. Next we show that there is no separating equilibrium. Without loss of generality, assume in equilibrium $x_1 \leq x_2 \leq x_3$. At least one inequality is strict since there is no pooling equilibrium. Suppose $x_1 < x_2$. Then player 1 can move closer to the left of x_2 and earn more. Suppose $x_2 < x_3$. Then player 3 can move closer to the right of x_2 and earn more. Hence there is no separating equilibrium. To conclude, no pure strategy Nash equilibrium exists when $n = 3$.

Let $e = (1/4, x_2, 3/4)$ where $1/4 \leq x_2 \leq 3/4$. To show that e is reactive, first note that there is no profitable deviation for player 2; he always gets 1/4 when he chooses $x_2 \in [1/4, 3/4]$, and he gets

[3] By efficient we mean that x_1, x_2 minimizes the average distance of the customers to the ice cream stands. Formally, (x_1, x_2) solves $\min_{x_1,x_2} \int_0^1 \min\{|x - x_1|, |x - x_2|\}dx$.

worse if he chooses points in $(0, 1/4) \cup (3/4, 1)$. For player 1, profitable deviations lie in $x \in (1/4, x_2)$. For any such move, player 2 can deviate to 1/4 and earn more than 1/4. Furthermore, if player 1 deviates back to $[0, 1/4]$, it will not make player 2 worse than before. Player 3 has no profitable deviation to $[0, 1/4]$, so no one will make player 2 worse off after player 2's deviation in response to player 1's. A symmetric argument can be shown for player 3. Hence e is reactive.

However, one can verify that the efficient location is $(1/6, 3/6, 5/6)$, which is not reactive. Player 1 can move closer to player 2. And the only agent that has another profitable deviation is player 3, which will also move closer to player 2, and this will not make player 1 worse off. So player 1 does not have to worry about the reactor at all.

(C) If $n = 4$, then (1/4,1/4,3/4,3/4) is a Nash equilibrium.[4] Each player makes 1/4 in this equilibrium. If a player located at 1/4, say, moves either to the left or the right then he makes less than 1/4. Similarly, a player at 3/4 cannot profit by moving.

The efficient location is $(1/8, 3/8, 5/8, 7/8)$. This is also not reactive for the reason same as the $n = 3$ case. For a reactive equilibrium, consider $e = (1/5, 2/5, 3/5, 4/5)$, observe that for player 2 and 3, there is no profitable deviation. For player 1, the only profitable deviation is to move a little bit right toward $x = 2/5$, but then player 3 can take $x = 1/5$ and make a profitable deviation. And there is no further profitable deviations of other players that can harm player 3 who is now in $x = 1/5$.

9.3 Signaling

Solution 9.3.1.

(A) The worker will earn $w - 4z^2/\theta_i$ for sure. Firms' offers depend their belief about worker types. Assume that firms have the same beliefs. If they believe that the worker is a high type, they earn $6 - w$, where w is the wage offer. If, instead, they assign probability p that the worker is of type 0, then each firm expects to earn

$$\Pi = 2p + (1 - p)6 - w.$$

(B) See Figure 2.

(C) Consider the following pooling-strategy profile and beliefs. Both workers choose $z = 0$, the firms believes that with probability 0.8 that the worker is of type 0. Thus, the expected marginal product

[4] An earlier version of the answers incorrectly stated that there is no Nash equilibrium when $n = 4$. We are grateful to Tom Struppeck of UT Austin for pointing out the error.

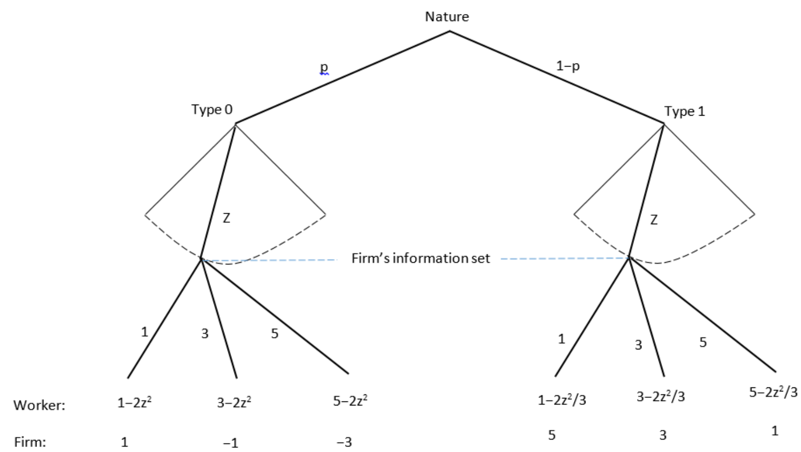

Figure 2: Ex 9.3.1(B)

of labor is $(0.8)2+(0.2)6 = 2.8$. The firms offer $w = 1$. This constitutes a Bayesian Nash equilibrium because both sides are taking best responses given the other side's action. Furthermore, firms' beliefs are consistent with the given actions.

There is a separating equilibria: Low type worker chooses $z = 0$, high type worker chooses $z = \sqrt{2}$. If a firm observes $z \geq \sqrt{2}$ the firm believes the worker is high type with probability one. Otherwise the firm believes the worker is of low type. Firms offer $w = 1$ to the low type and $w = 5$ to the high type. Since

$$1 = 5 - \frac{4(\sqrt{2})^2}{2},$$

low type has no incentive to deviate. Similarly, $5 - 4(\sqrt{2})^2/6 > 1$ so high type also does not deviate. The firms' beliefs are consistent with the actions of the worker, and if a firm deviates, it either make a loss on the low workers or loses the chances to hire a high worker. So this is also a Bayesian Nash equilibrium.

(D) The pooling equilibrium considered in (C) does not satisfy the weak communication test. The high type should deviate to $z = \sqrt{2} + \epsilon$ and claim that he should be believed as a high type and receive $w = 5$, since the low type will prefer to stay at $(z, w) = (0, 1)$ instead. It also follows that it fails the strong communication test.

The separating equilibria satisfies both tests, since the pooling wage is still 1, the high type does not have an incentive to be pooled.

(E) The separating equilibrium proposed in (C) remains an equilibrium since it is independent from proportions. For the pooling equilibria, the expected marginal product of labor when workers are pooled is now $(0.5)2 + (0.5)6 = 4$.

Pooling equilibrium 1: Both types of workers choose $z = 0$, firms believe the worker is of high type with probability 0.5 and offer $w = 3$.

Pooling equilibrium 2: Both types of workers choose $z = 1$, firms believe the worker is of low type when $z < 1$ and believe the worker is high with probability 0.5 if $z \geq 1$. The firm offers $w = 1$ if $z < 1$ and $w = 3$ if $z \geq 1$.

(F) The separating equilibrium satisfies the weak test because the low type is already indifferent between the high type's (z, w) pair and his own $(0, 1)$. So the high type can not claim to be high if he chooses $z < \sqrt{2}$.

For pooling equilibrium 1, the high type can choose z slightly larger than 1 and claim he is high type and should be paid $w = 5$ because a low type doing so will be worse off. Hence it fails the weak test.

For pooling equilibrium 2, the high type can choose z slightly larger than $\sqrt{2}$ and claim to be a high type and should be paid $w = 5$ because the low type doing so will be worse off. Hence it fails the weak test.

(G) Because the pooling equilibria fail the weak test, they fail the strong test. For the separating equilibrium, the high type does not want to be pooled either, since the best thing he can earn in a pooling equilibrium is $(z, w) = (0, 3)$, which gives lower utility than $(z, w) = (\sqrt{2}, 5)$. Hence it satisfies the strong test.

Solution 9.3.2.

(A) For any $z \in [z_1, z_2]$, let w be such that (z, w) lies on the indifference curve U_2^S. Then any $w' \in [w, \theta_2]$ benefits type 2. Since they also lie above the indifference curve U_1^S, they also benefit type 1. Hence for any $z \in [z_1, z_2]$, the set of wages that benefits type 1 contains the set of wages that benefits type 2.

(B) Note that the divine equilibrium adds an additional requirement to the out-of-equilibrium-path beliefs of a sequential equilibrium. For the proposed sequential equilibrium, let the beliefs be that when ever the firms observe $z \in [0, z_1)$, the firms believe the type is 0, whenever the firm sees $z \in [z_1, z_2)$, the firms believe the type is 1. Else the firm believes the types to be 2.

This belief clearly satisfies the divine criterion because the set of wages that benefit the higher types also benefits the lower types, so the firm should be more inclined to believe the type is lower.

(C) No, because given such situation the firm should not believe that it is more likely to be type 1 that makes the deviation, since type 2 workers are a lot more than type 1 workers, even if, conditional on types, type 1 is more likely to make the deviation, the overall probability that the deviation is made by a type 2 worker is still very high.