The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 8: Informational Asymmetry and Contract Design

8.1 Hidden Actions (Moral Hazard) and Contract Design

Solution 8.1.1.

(A) Suppose $\hat{k}(H) > k(H)$, then given the same Z the agent needs a higher R to induce high effort. Hence the BB' curve in the RZ diagram shifts outward. Also, the indifference curve of the agent when he chooses high effort also shifts upward(relative to the low effort reservation utility). The principle's indifference curve remains unchanged. Intuitively, this makes the contract that induces high effort more costly for the principle. The optimal R^* and Z^* will increase, in a situation where the high effort inducing contract is optimal. (Note that the kink of the agent's indifference curve, which is on the new BB' curve, is to the upper right of the original kink.)

(B) Eventually as $\hat{k}(H)$ becomes too high, it will be too costly for the principal to offer a contract that induces high effort, thus the optimal contract will reduce to $(0, R^{**})$.

Solution 8.1.2.

(A) If the firms are competitive, the expected profit of each firm will be driven to zero in equilibrium. Hence the contract will lie at the firm's indifference curve that gives zero profit, which passes the point $(Z, R) = (y_2 - y_1, y_2)$. The agent is then able to get an utility higher than his reservation utility.

(B) If there are many agents, then the contract will stay at the agent's indifference curve with reservation utility $\overline{V}$. But this will be the same as the base-line model.

(C) Since the supply of agents is perfectly elastic at $\overline{V}$ and firms choose contracts, each firm can maintain the contract that will be proposed in (B). As an analogy, if labor supply is perfectly elastic, no matter what the demand of labor, the equilibrium price will be the same.

Solution 8.1.3.

(A) Assume $\phi(x)$ is strictly increasing. For any $h > 0$, simply choose $\beta = h$. Then if the agent deviates to $x < x^*$, there will be a positive probability (at least 0.5) that $\tilde{\epsilon}\phi(x) < \epsilon_1\phi(x^*)$. But then the expected utility for the agent to choose x will be negative infinite. Hence the agent will choose x^* so that he can avoid the penalty with probability one.

(B) As long as $\phi(x)$ is strictly increasing, the lower support of the output distribution $\tilde{\epsilon}$ is strictly increasing in x. So if the agent takes the action $x*$, she will not be penalized by an incentive scheme that pays h for all $y \geq \epsilon_1\phi(x^*)$. On the other hand, if she chooses $x < x*$ the probability of an output $y < \epsilon_1\phi(x^*)$ is strictly positive. By making the penalty sufficiently large, the agent will be discouraged from any finite deviation $x < x^*$.

(C) If ϵ_1 is zero the lower support of the output distribution is zero. Any penalty scheme must then punish, with positive probability, even an agent who takes the correct action.

(D) In practice it would be difficult for the principal to convince the agent about the lower support of the distribution. Any doubts of this kind would make the agent unwilling to accept a contract that involves even a very small chance of unbounded losses. Of course, in actuality unbounded losses cannot be imposed as the agent has the option of declaring bankruptcy.

Solution 8.1.4.

(A) First note that for each x the agent chooses, the principal can adjust h so that the agent receives reservation utility. Hence it optimum:

$$\overline{U}_A = h + sy(x) - K(x) - \frac{1}{2}\alpha_A s^2\sigma^2. \tag{1}$$

The principal receives

$$U_P(s, x, h) = (1-s)y(x) - h - \frac{1}{2}\alpha_P(1-s)^2\sigma^2.$$

Plugging in (1) to obtain

$$\begin{aligned} U_P(s, x, \overline{U}_A) &= (1-s)y(x) - \overline{U}_A + sy(x) - K(x) - \frac{1}{2}\alpha_A s^2\sigma^2 - \frac{1}{2}\alpha_P(1-s)^2\sigma^2 \\ &= y(x) - K(x) - \frac{1}{2}\alpha_A s^2\sigma^2 - \frac{1}{2}\alpha_P(1-s)^2\sigma^2 - \overline{U}_A. \end{aligned}$$

(B) Note that $\overline{U}_A$ enters linearly into the principal's utility, hence it does not affect the FOCs of s and x. The only thing in the contract that changes with $\overline{U}_A$ is h.

(C) If the effort level is observable, taking FOC w.r.t. x and s in the principal's utility to obtain

$$x: \quad y'(x) = K'(x)$$
$$S: \quad \sigma^2(-s\alpha_A + (1-s)\alpha_P) = 0.$$

Solving the second equation for s to get

$$s = \frac{\alpha_P}{\alpha_A + \alpha_P}$$

(D) Suppose that effort is unobservable, then given a contract (h, s) the agent will choose x to maximize his expected utility. Taking FOC of (1) w.r.t. x to get

$$sy'(x(s)) - K'(x(s)) = 0. \tag{2}$$

Note that the optimal $x(s)$ will in general depend on s.

(E) Differentiate (2) w.r.t. s to obtain

$$y'(x(s)) + sy''(x(s))x'(s) - K''(x(s)))x'(s) = 0$$

So

$$x'(s) = \frac{y'(x(s))}{K''(x(s)) - y''(x(s))} > 0$$

if and only if

$$K''(x(s)) - y''(x(s)) > 0$$

since y is increasing.

(F) Plugging $x(s)$ into U_P and taking FOC w.r.t. s to obtain

$$\frac{dU_P}{ds} = x'(s)(y'(x(s)) - k'(x(s))) - \sigma^2(s\alpha_P - (1-s)\alpha_a).$$

When $s \leq s^* < 1$, the second term(including the minus sign) is positive, and the first term is positive since $x'(s) > 0$ and $y'(x(s)) - K'(x(s)) > sy'(x(s)) - K'(x(s)) = 0$. Hence $dU_P/ds > 0$.

When $s = 1$, the second term of dU_P/ds becomes negative, and the first term is zero by (D).

(G) By (F) the sharing rule when effort is unobservable will be larger than s^*. Intuitively, when the principal can not monitor the agent, he needs to provide more incentive for the agent to exert high effort.

8.2 Hidden Knowledge

8.2.1 Adverse Selection

Solution 8.2.1.1.

(A) If the price for used car is p, only the cars with value less than p will be sold. Hence the average value of the used car conditional on $v < p$ is $1 \times \mathrm{E}[v|v < p] = (\alpha + p)/2$ since used cars last only one period.

(B) Suppose the equilibrium average value of the used car is v and price is p. Then $v = (\alpha + p)/2$. But then the willingness to pay will also be $p = v$. Hence, in equilibrium, only the car with value $v = \alpha$ will be traded at a price $p = \alpha$.

(C) In the new car market, the buyer expects not be able to sell his car in the next period, hence the expected value of the new car is $2 \times \mathrm{E}[v] = \alpha + \beta$.

(D) Virtually no cars will be traded in the used car market. With more detailed setting about the utility functions of buyers and sellers, there could be some efficiency loss.

(E) The asymmetric information problem still exists, and if one can not distinguish used cars and new cars in the market for new cars, new cars will be driven out from the market and there will only be used cars.

Solution 8.2.1.2.

(A) Note that the reservation utility is increasing with θ, hence given an equilibrium wage w^*, there should be a cutoff θ^* such that agents with $\theta > \theta^*$ chooses self-employment, and any $\theta < \theta^*$ chooses to be hired.

The equilibrium (w^*, θ^*) satisfies

$$\begin{aligned}(1-d)\theta^* &= w^* \\ w^* &= \frac{\alpha + \theta^*}{2}.\end{aligned}$$

Hence we can solve for θ^* to obtain

$$\theta^* = \frac{\alpha}{2(1-d) - 1} \tag{3}$$

for appropriate ranges of d.(When d is large enough every agent will choose to be employed)

Note also that when $d < 1/2$, smaller d implies smaller θ^*. Plug in $d^* = 1/2(1+\alpha)$ into (3) we obtain

$$\theta^* = 1 + \alpha.$$

Hence for $d < d^*$ the agent with highest marginal productivity will remain self-employed.

(B) Since agents with MPL $\theta^* < \theta < 1 + \alpha$ remain self-employed. It follows from (3) that the proportion $P(d, \alpha)$ is

$$P(d, \alpha) = \frac{1 + \alpha - \theta^*}{1 + \alpha - 1} = 1 + \alpha - \frac{\alpha}{2(1-d) - 1}.$$

(C) One can then readily observe from (B) that $\lim_{d \to 0} P(d, \alpha) = 1$, which means that approximately every agent will choose to be self-employed.

(D) Suppose $(1 - \mu)\lambda + \mu\theta < \theta$ for all $\theta \in [\alpha, \alpha + 1]$. Plugging in θ to obtain

$$(1 - \mu)\lambda + \mu\alpha < \alpha.$$

Rearrange to get

$$\lambda < \alpha.$$

Similar to (A), one can also solve the equilibrium θ^* by

$$(1 - \mu)\lambda + \mu\theta^* = \frac{\alpha + \theta^*}{2}.$$

Assuming $\lambda < \alpha$, then the smaller μ is, the worse self employment is. Hence we only need to check whether there is adverse selection when $\mu = 1/2$. Plugging $\mu = 1/2$ into the equilibrium equation above we obtain

$$\frac{\lambda + \theta^*}{2} < \frac{\alpha + \theta^*}{2},$$

hence everyone will choose to be employed.

In the limiting case where $\mu = 1$, self-employment delivers the agent his full MPL θ, while the market will bring down his salary because there are other agents with lower MPL in the labor market. so everyone except the agent with the lowest MPL chooses self-employment.

Solution 8.2.1.3.

(A) From Figure 1 (the blue lines) one can see that the bank's return $R_b(y)$ is concave and the entrepreneur's $R_e(y)$ is convex.

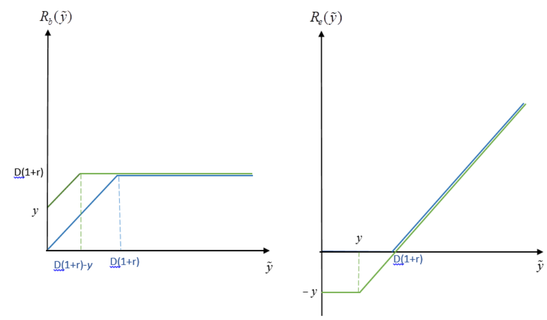

Figure 1: Ex 8.2.1.3

(B) Since $\tilde{y}_A \succ \tilde{y}_B \succ \tilde{y}_C$ in terms of second order stochastic dominance and the bank's utility function is concave and the entrepreneur's is convex, we have

$$E[R_b(\tilde{y}_A)] > E[R_b(\tilde{y}_B)] > E[R_b(\tilde{y}_C)],$$

$$E[R_e(\tilde{y}_A)] < E[R_e(\tilde{y}_B)] < E[R_e(\tilde{y}_C)].$$

Bank prefers to fund project A since the expected utility of $\tilde{y}_A$ to the bank is the highest. The entrepreneur prefers $\tilde{y}_C$. If the bank cannot distinguish between the three projects, it will end up with $\tilde{y}_A$.

(C) When the agent defaults, he loses his collateral, y, and all his return up to $D(1+r) - y$. Then from Figure 1 (green lines) we see that $R_b(y)$ is concave and $R_e(y)$ is convex. Therefore, the answers to (A) and (B) do not change.

Solution 8.2.1.4.

(A) Assume that the Maori will match if and only if $B < 2N/3$, then the visitor gets

$$v(B) = (N - B)1_{\{N \leq 3B/2\}},$$

where 1_S is the indicator function. So the visitor's expected utility is

$$\begin{aligned} E[v(B)] &= E[(N-B)1_{\{N<3B/2\}}] \\ &= \frac{1}{100} \sum_{N=0}^{\min\{\lfloor \frac{3}{2}B \rfloor, 99\}} N - B \end{aligned}$$

One can observe that $E[v(0)] = 0$ and $E[v(B)] < 0$ for all B, so the visitor will bid $B = 0$.[1]

[1] A way to see this is to approximate the expected utility with the continuous version

$$E[v(B)] \approx \frac{1}{100}\left(\frac{(3B/2)(3B/2+1)}{2} - (\frac{3B}{2}+1)B\right).$$

(B) The expected payoff to the visitor is 0, and to the Maori is $E[2N/3] = 33$.

(C) The expected utility is then

$$E[v(B)] = \frac{1}{100} \sum_{N=0}^{\min\{\lfloor \frac{1}{k}B \rfloor, 99\}} N - B.$$

The FOC w.r.t. B in the continuous approximation is

$$\frac{\partial E[v(B)]}{\partial B} \approx \frac{B}{k^2} + \frac{1}{2k} - \frac{2B}{k} - 1 = 0.$$

Hence when k is sufficiently small, bidding $B \geq 1$ is better than bidding $B = 0$ for the visitor.

8.2.2 Screening

Solution 8.2.2.1.

(A) Suppose there are three types, with type 1 being the one with lowest risk, and type 3 the highest risk. Their indifference curves on the (Z, R) plane is depicted as Figure 2.

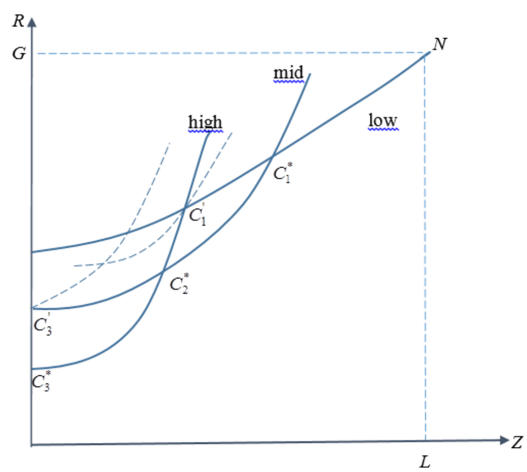

Figure 2: Ex 8.2.2.1

(B) The optimal pricing rule will pool together the middle risk agent with the low income group. This is because a pricing plan that makes type 1 and type 2 closer in Figure 2 will make C_1^* downward (along 1's indifference curve all the way to C_1'), hence earn profit from type 1, while sacrificing some of the profits from type 2. On the other hand, a pricing plan that makes type 2 and type 3 closer will lose profit from type 3 and earn extra profit from type 2(all the way to C_3'). Since the proportion of type 2 agents is small, clearly the former is better than the latter.

(C) From Figure 2 we can conclude

- (i) The policy extracts the surplus of the agents with lowest risk.(He is indifferent between insure or not)
- (ii) The policy fully covers the agents with the highest risk.(His deductable is zero)
- (iii) The higher the risk the lower the deductible.

Actually, item (iv) in the text can also be carried over to the n-type model. Hence everything in the proposition generalizes to the n-type model.

Solution 8.2.2.2.

(A) The expected utility for type t agent is

$$U_t(Z,R) = \sum_{s\in S} \pi_s^t v(w-G+R-Z) + (1-\sum_{s\in S}\pi_s^t)v(w-G+R),$$

so the slope of his indifference curve is

$$\frac{dR}{dZ} = \frac{\sum_{s\in S}\pi_s^t v'(w-G+R-z)}{\sum_{s\in S}\pi_s^t v'(w-G+R-z) + (1-\sum_{s\in S}\pi_s^t)v'(w-G+R)}.$$

The principal's expected profit when he offers (Z,R) to type t is

$$H_\pi(Z,R) = G-R-\sum_{s\in S}\pi_s^t(L_s-Z),$$

so the slope of his indifference curve is

$$\frac{dR}{dZ} = -\sum_{s\in S}\pi_s^t.$$

Let $\pi_t = \sum_{s\in S}\pi_s^t$. Then the assumption that $\pi_s^1 < \pi_s^2$ for all s implies $\pi_1 < \pi_2$. Hence type 1 is the low risk type and type 2 the high risk type.

(B) By the above computation about the indifference curves, only the overall probability of a claim matters.

(C) He might try to make the deductible Z state contingent.

8.2.3 Monopoly Price Discrimination with Hidden Knowledge

Solution 8.2.3.1.

(A) With a quantity constraint, the seller will be maximizing the objective function given in (8.2.9) with a quantity constraint. Specifically, he solves

$$\max_{q_1,q_2} f_1B_1(q_1) - f_2(B_2(q_1) - B_1(q_1)) + f_2B_2(q_2)$$

s.t.

$$q_1 + q_2 = Q.$$

The FOCs are then

$$f_2B_2'(q_2) = \lambda$$
$$f_1B_1'(q_1) - f_2(B_2'(q_1) - B_1'(q_1)) = \lambda.$$

(B) The shadow price is the increment in revenue when Q is increased by a little, or just the marginal revenue.

(C) If marginal revenue(λ) is larger than marginal cost $C'(q_1 + q_2)$, the firm should produce more. In optimum we should have MR=MC.

Solution 8.2.3.2.

(A) Since $B_1'(z) < B_2'(z)$, $z_1^* < z_2^*$. The reservation indifference curves (zero utility) are plotted in Figure 3. Positive utility indifference curves are obtained by shifting these indifference curves down vertically.

(B) The reservation indifference curve will be the curve on which the utility is zero, hence it is the curve

$$\pi = B_i(z) - C(z).$$

(C) If types are distinguishable, the firm will maximize profit π separately, namely, the firm chooses z_i for buyer i such that

$$B_i'(z_i) - C'(z_i) = 0,$$

which is z_i^* shown in Figure 3. The firm charges $p_i^* = B_i(z_i^*) - C(z_i^*)$, that is, the firm extracts full surplus for each of the buyers.

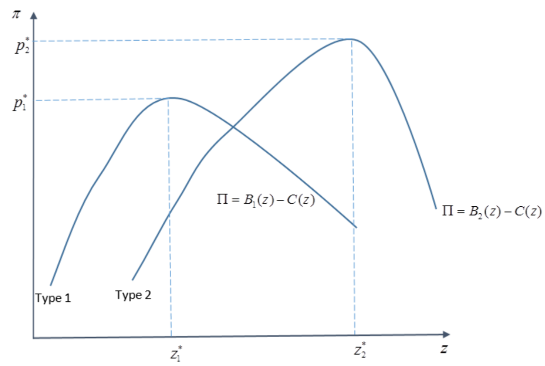

Figure 3: Ex 8.2.3.2

(D) Given a contract under which the low type (who values quality less) receives positive surplus, the firm can fix (z_1, z_2) but increase the prices charged to the two consumers by the same amount until the low type gets zero utility. This adjustment does not change the incentive compatibility of the contract but increases profit. More over, incentive compatibility implies high type's individual rationality, hence under such adjustment the high type will indeed still want to buy.

Assume the proportion of the two types is (f_1, f_2). Let $(z_1, p_1), (z_2, p_2)$ denote the contract offered by the firm. Then the total profit is given by (8.2.9) minus the costs:

$$\Pi = f_1 B_1(z_1) - f_2(B_2(z_1) - B_1(z_1)) + f_2 B_2(z_2) - f_1 C(z_1) - f_2 C(z_2).$$

The FOCs are then

$$f_1 B_1'(z_1) - f_2(B_2'(z_1) - B_1'(z_1)) - f_1 C'(z_1) = 0$$
$$B_2'(z_2) = C_2'(z_2).$$

Suppose $z_1 = z_2$, then the first FOC implies

$$B_1'(z_1) - C'(z_1) = 0.$$

But this will contradict the second FOC, since $B_2'(z) > B_1'(z)$ for every z. Hence in optimum the two types receive goods with different quality.

(E) Intuitively, when the firm only has a single bundle, since the marginal utility of the high type is higher, he can keep the profits from the low type constant while increase the profits from the high type by decreasing the quality and price for the low type and increase the quality and price for the high type. It will be incentive compatible since the marginal utility of quality for the high type is higher, so he is willing to pay the extra price, while the low type will not willing to buy the high type's bundle.

(F) Mathematically the two models are very similar, in that the firm maximizes an objective function subject to incentive compatibility constraints and individual rationality constraints. However, in the insurance model, when the good risk type is sufficiently low, they will not receive insurance. But in the current model, it is always optimal to separate the two types.

Solution 8.2.3.3.

(A) If the monopolist receives \$T when he sells q units his profit is:

$$\Pi = T - cq.$$

Therefore, for any schedule of (q, T) pairs, there is a schedule of (q, Π) pairs. A type i buyer has a consumer surplus of

$$\begin{aligned} U_i &= \int_0^q p_i(x)dx - T = \int_0^q p_i(x)dx - cq - \Pi \\ &= \int_0^q p_i(x) - cdx - \Pi. \end{aligned}$$

(B) Let $B_i(q) = \int_0^q p_i(x)dx$ and $C(q) = cq$, then mathematically we get the same problem as in Exercise 2. Type 1 agent will get zero utility, hence

$$\Pi_1 = \int_0^{q_1} p_1(x) - cdx.$$

Type 2 agent will get a q_2 that maximizes $B_2(q_2) - C(q_2)$, or, in terms of FOC,

$$p_2(q_2) = c.$$

Type 2 is also indifferent between (q_1, Π_1) and (q_2, Π_2), so we can solve for

$$\Pi_2 = \int_0^{q_2} p_2(x) - cdx - \int_0^{q_1} p_2(x) - cdx + \int_0^{q_1} p_1(x) - cdx$$

The firm then solves

$$\max_{q_1} f_1\Pi_1 + f_2\Pi_2.$$

(C) For the numerical example:

$$\begin{aligned} U_1 &= \int_0^q (8 - x)dx = 8q - \frac{q^2}{2} - \Pi \\ U_2 &= \int_0^q (8 - \frac{x}{2})dx = 8q - \frac{q^2}{4} - \Pi. \end{aligned}$$

Using (B), we get

$$\Pi_1 = 8q_1 - \frac{q_1^2}{2}.$$

Type 2's indifference curve is

$$\Pi_2 q = 8 - \frac{q^2}{4} - U_2,$$

the FOC implies

$$q_2^* = 16.$$

Type 2's incentive compatibility constraint then gives

$$8q_1 - \frac{q_1^2}{4} - (8q_1 - \frac{q_1^2}{2}) = 8(16) - \frac{16^2}{4} - \Pi_2,$$

hence

$$\Pi_2 = 64 - \frac{q_1^2}{4}.$$

(D) Assume the proportion of type 1 is f. The firm then solves

$$\max_{q_1} f(8q_1 - \frac{q_1^2}{2}) + (1-f)(64 - \frac{q_1^2}{4}).$$

The FOC is

$$f(8 - q_1^*) - (1-f)\frac{q_1^*}{2} = 0.$$

When $f = 0.5$, $q_1^* = 16/3 > 4$.

(E) More generally, $q_1 < 8$ for all $f < 1$, and so

$$\frac{T_1}{q_1} = 10 - q_1/2 > 6.$$

On the other hand:

$$\frac{T_2}{q_2} = \frac{64 - q_1^2/4 + 2q_1}{16} = \frac{64 - (8 - q_1)q_1/4}{16} < 4 \text{ for all } q_1 < 8.$$

Therefore, quantity discounting prevails for all f.

(F) It is most applicable for commodities where resale is costly, either because of the nature of the product(e.g., electricity) or because the seller has legal protection.

Solution 8.2.3.4.

(A) Recall that from the discussion in the last paragraph of this section, where we have three types, the FOC for q_1 is given by

$$f_1 p_1(q_1) - (f_2 + f_3)[p_2(q_1) - p_1(q_1)] = 0.$$

Plugging in the demand functions we get

$$f_1(11 - q_1) - (f_2 + f_3) = 0.$$

Hence we need

$$f_1 > \frac{f_2 + f_3}{11}.$$

In such cases,

$$q_1 = 11 - \frac{f_2 + f_3}{f_1}$$

(B) Now recall the FOC for q_2:

$$f_2 p_2(q_2) - f_3[p_3(q_2) - p_2(q_2)] = 0.$$

Plugging in the demand functions we get

$$f_2(12 - q_2) - f_3 = 0,$$

or

$$q_2 = 12 - \frac{f_3}{f_2}$$

For the highest type we know that they will get the efficient level of q, the FOC is then

$$p_3(q_3) = 0.$$

Plug in the demand function to get

$$q_3 = 13.$$

Since $q_2 < q_3 = 13$, we only need to ensure $q_1 < q_2$. In terms of the parameters, we need

$$11 - \frac{f_2 + f_3}{f_1} < 12 - \frac{f_3}{f_2}.$$