The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 7: Strategic Uncertainty and Equilibrium Concepts

7.1 Dominant Strategy

This section contains no exercises.

7.2 Nash Equilibrium

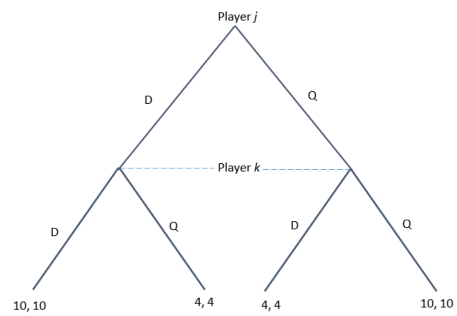

Figure 1: Ex 7.2.1

Solution 7.2.1. There are two pure strategy NEs: $(D, D), (Q, Q)$. To compute mixed NE, suppose player k plays D with probability q. Then for player j we must have

$$10q + 4(1 - q) = 4q + 6(1 - q),$$

or $q = 1/4$. By symmetry, the mixed NE is then $(p, q) = (1/4, 1/4)$.

7.3 Subgame-perfect Equilibrium

Solution 7.3.1.

(A) In the subgame where the entrant has chosen enter, the incumbent's optimal action is Match. Given this, the entrant will choose enter. Hence (Enter, Match) is a SPNE.

Another NE is (Out, Undercut), but in this case the incumbent is not acting optimally at his off-the-equilibrium-path decision node where the Entrant enters. Hence this is not a SPNE.

(B) If $\pi \in (0, 1)$, then we have a game tree that has no proper subgame. Hence every NE is automatically a SPNE. One can check that the NEs are ((Out,Out), Undercut) and ((Enter,Enter), Match).

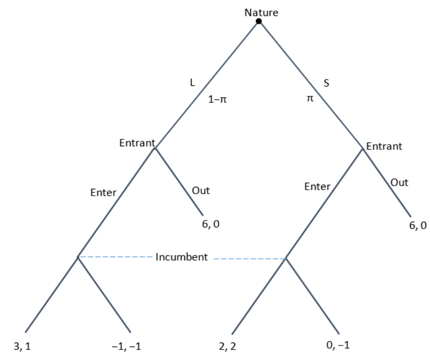

Figure 2: Ex 7.3.1(B)

(C) Then the NE ((Out,Out), Undercut) is ruled out. If the entrant trembles to play Enter, i.e., the incumbent's information set is reached with positive probability, then the optimal response of the incumbent is to Match.

(D) In both equilibria, the entrant's strategies are optimal response against small trembles of the incumbent. For ((Out,Out), Undercut), if there is just a ϵ probability that the incumbent chooses Match, then the expected utility of Out is still strictly larger than Enter. For ((Enter,Enter), Match), if there is only a slight probability that the incumbent chooses Undercut, the expected utility of Enter is still strictly greater than Out. Thus, the answer does not change.

Solution 7.3.2.

(A) If Alex bids 3500, a deviation by Bev (from bidding 2500) divides into two cases: $b_B < 3500$ or $b_B \geq 3500$. In the first case Bev's utility is the same as bidding 2500 as she will not win with the original bid of 2500 or with a deviation to any bid less than 3500. In the second case Bev will win and pay 3500 or more (with probability 0.5 if she bids exactly 3500, with probability 1 if she bids more than 3500); but then her utility will be negative. Hence bidding 2500 is a best response. A similar argument holds for Alex.

(B) Given $b_A = 0$, Bev will win by bidding any $b_B > 0$ and she only pays 0. So $b_B = 10000$ is a best response. Given $b_B = 10000$, Alex will lose the auction with any bid less than 10000. Alex wins only if he bids 10000 or more but that makes Alex worse off; he pays 10000 for an item worth 3500 to him. Hence $b_A = 0$ is a best response for Alex. There are lots of equilibria of this type, where one bidder bids very low (less than any of the bidders' values) and the other bids very high (more than the highest of the two bidders' values).

(C) If Bev bids $b_B > 2500$, an amount greater than her value of 2500, then she will win and make a loss if Alex bids an amount between 2500 and b_B. If Alex bids less than 2500 or more than b_B, then Bev obtains the same payoff whether she bids her value 2500 or b_B. Thus bidding more than her value is weakly dominated by bidding her value. If, on the other hand, Bev bids less than her value, $b_B < 2500$ and Alex bids an amount between b_B and 2500, Bev will not win; however, she would win and make a positive profit with of bid of 2500. If Alex bids less than b_B or more than 2500, Bev gets the same profit with b_B or 2500. Thus, bidding 2500 weakly dominates bidding less than 2500. A similar argument applies to Alex – a bid equal to his value of 3500 dominates any other bid for Alex.

Hence the equilibrium of (A) is more credible than any other equilibrium.

7.4 Further Refinements

Solution 7.4.1.

(A) Given player 1,2 play the completely mixed strategies (p, q), where p, q are the probabilities of playing Up. Player 3's belief in his info set is then

$$P(\text{Upper}) = \frac{p}{p + (1-p)q}.$$

Hence the belief $P(\text{Upper}) = 1$ is the limit of a sequence of beliefs induced by completely mixed strategies where $p \to 1$ and $q \to 0$, along which player 1,2's strategies converge to (Up, Up).

One can also check that (Up, Up, Down) is sequentially optimal given the consistent belief $(1, 0)$. Hence it can be supported as a sequential equilibrium.

(B) Yes. Consider the sequence of completely mixed strategy (p, q) with $p \to 0$ and $q \to 1$. Then $P(\text{Upper}) \to 0$. So the belief $(0, 1)$ is consistent. Furthermore, under such belief (Down, Up, Up) is sequentially optimal. Hence it's also a sequential equilibrium.

(C) Suppose player 3 mixes in equilibrium. Then player 3 must be indifferent between Up and Down. Given the belief on his information set be $(\pi, 1-\pi)$, we must have

$$\mathrm{E}\,U_3(\text{Up}) = 2 - \pi = \mathrm{E}\,U_3(\text{Down}) = 1 + \pi.$$

So $\pi = 1/2$. To formulate such belief under Bayesian updating, player 1,2, must play completely mixed strategies (p, q) such that

$$\frac{p}{p + (1-p)q} = \frac{1}{2}.$$

Hence they are also indifferent between their actions. Suppose player 3 plays $(r, 1-r)$ in equilibrium. Then we have

$$\mathrm{E}\, U_2(\text{down}) = \frac{1}{2} = \mathrm{E}\, U_2(\text{Up}) = 4r - 1,$$

which means $r = 3/8$. For player 1, we then have

$$\mathrm{E}\, U_1(\text{Down}) = (1-q)\frac{1}{2} \leq \frac{1}{2}$$
$$\mathrm{E}\, U_1(\text{Up}) = -1 \times \frac{3}{8} + 3 \times \frac{5}{8} = \frac{3}{2} > 1,$$

so player 1 does not have an incentive to mix. Hence there is no sequential equilibrium in which player 3 mixes.

Solution 7.4.2.

(A) Given (Up, q, Up), it is optimal for player 3 to choose up since whenever his information set is reached it is at the upper node. Given $q < 2/3$ and player 1 chooses up,

$$\mathrm{E}\, U_1(\text{Up}, q, \text{Up}) = 3 > \mathrm{E}\, U_1(\text{Down}, q\text{Up}) = 4q + (1-q),$$

so player 1 will also choose Up. Given player 1 choose Up, player 2 gets the same utility no matter what his strategy is since his decision node will not be reached. Hence (Up,q,Up) is a Nash equilibrium.

Given (Down, Down, r) where $r \leq 1/3$, first note that player 3's strategy is optimal since his information set will not be reached. For player 2,

$$\mathrm{E}\, U_2(\text{Down}, \text{Down}, r) = 2 > \mathrm{E}\, U_2(\text{Down}, \text{Up}, r) = 4r.$$

So it is optimal for player 2 to choose Down. For player 1,

$$\mathrm{E}\, U_1(\text{Down}, \text{Down}, r) = 1 > \mathrm{E}\, U_1(\text{Up}, \text{Down}, r) = 3r,$$

so player 1 will also choose Down. Hence (Down,Down,r) is a NE.

(B) The equilibrium (Up, q, Up) where $q < 2/3$ is not sequential. The reason is that conditional on player 2's decision node is reached, it is optimal for player 2 to choose Up with probability 1 rather than $q < 1$ given player 3 chooses Up.

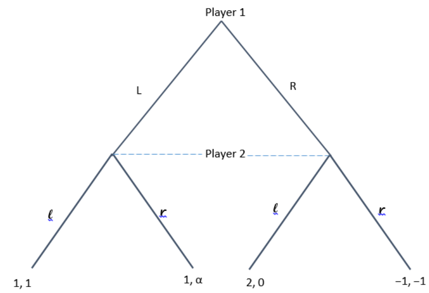

Figure 3: Ex 7.4.3(B)

The equilibrium (Down, Down, r), with player 3's off-the-equilibrium path belief $(1/3, 2/3)$, is a sequential equilibrium. One can check consistency by considering the sequence of completely mixed strategies of player 1 and 2 given by $(p, q) = (\epsilon, 2\epsilon)$. Under such strategy the belief converges to $(1/3, 2/3)$ as $\epsilon \to 0$, and under such belief player 3 is indifferent between his actions once his information set is reached. The sequential optimality for the rest of the players is already checked in (A).

Solution 7.4.3.

(A) A trembling hand equilibrium requires that each strategy is a best response against a small enough tremble of opponents' strategies. Since a tremble is completely mixed, the continuity of expected utility with respect to beliefs and mixed strategies implies sequential rationality. The limiting belief is consistent by definition.

(B) See figure.

(C) One can easily check that the pure strategy NEs are (R, l) and (L, r). In each equilibrium, player 2's strategy is a strict best response if $\alpha > 1$. Hence for small tremble ϵ_1 by player 1, player 2's strategy remains a strict best response in each NE.

(D) Similarly, in the given NEs player 1's strategy is a strict best response. Hence for small ϵ_2 they remain strict best responses.

(E) When $\alpha = 1$, strategy r is weakly dominated by l. Hence, whenever player 1 puts positive probability on R, player 2 should play l. So only the equilibrium (R, l) is trembling hand perfect.

Solution 7.4.4.

(A) This is a game of perfect information. Each information set has a single node and one need not concern oneself with beliefs at an information set. Thus, any subgame-perfect Nash equilibrium is a sequential equilibrium.

(B) Player 2 rejects at B_1 is not trembling hand perfect. If player 1 trembles to reject at A_2, player will gain by accepting at B_1.

(C) If the parameters V_A, V_B are slightly changed, the conclusion is still the same. Because there are still huge room between the bids(in thousands) and the true values.

(D) It is because we have an extensive form game here, whereas in in the previous exercise we had a normal form game. In normal form games, most sequential equilibria are also trembling hand perfect.

7.5 Games With Private Information

Solution 7.5.1.

(A) The joint distribution matrix of (v_a, v_b) is given by

$$\begin{pmatrix} \frac{1}{4} & \beta \\ \beta & \frac{3}{4} - 2\beta \end{pmatrix}$$

So $E[v_a] = E[v_b] = 1/4 + \beta - 4(3/4 - \beta) = 5\beta - 11/4$. Then

$$\begin{aligned} \mathrm{Cov}(v_a, v_b) &= E[v_a v_b] - E[v_a]E[v_b] \\ &= \frac{1}{4} - 8\beta + 16(1 - 2\beta - \frac{1}{4}) - (5\beta - \frac{11}{4})^2 \qquad = 12 + \frac{1}{4} - 40\beta - (5\beta - \frac{11}{4})^2. \end{aligned}$$

Substituting $\beta = 1/4$ we obtain $\mathrm{Cov}(v_a, v_b) = 0$, and the derivative of the covariance with respect to β is negative. Hence $\mathrm{Cov}(v_a, v_b) > 0$ for all $\beta < 1/4$.

(B) Suppose Bev follow the proposed strategy, denoted by s. For Alex, when his $v_a = 1$, his posterior belief about Bev's values is

$$P(v_b = 1 | v_a = 1) = \frac{1/4}{1/4 + \beta}$$
$$P(v_b = 4 | v_a = 1) = \frac{\beta}{1/4 + \beta}.$$

Letting A denote Aggressive and P Passive, we then have

$$\mathrm{E}[U_A(A, s) | v_a = 1] = 1 \times \frac{1/4}{1/4 + \beta} + 6\frac{\beta}{1/4 + \beta} > \mathrm{E}[U_A(P, s) | v_a = 1] = 3\frac{\beta}{1/4 + \beta}.$$

For Alex with $v_a = -4$ we have

$$\begin{aligned} \mathrm{E}[U_A(P,s)] &= 0 \times \frac{\beta}{3/4-\beta} + 3 \times \frac{3/4-2\beta}{3/4-\beta} \\ &> -4 \times \frac{\beta}{3/4-\beta} + 6\frac{3/4-2\beta}{3/4-\beta} \\ &= \mathrm{E}[U_A(A,s)] \end{aligned}$$

whenever $\beta > 9/40$. A symmetric argument shows that when Alex follows s, it is a best response for Bev to follow s. Hence, when $\beta > 9/40$ the proposed equilibrium is a BNE.

(C) When $\beta < 9/40$, the strategy profile will not be a BNE because type $v_i = -4$ will want to deviate, as shown in (B).

(D) The BNE here is a partially mixed one. Intuitively, if values are correlated, then it can not be true that the low value type choose the same action, while for a high value type Aggressive is always dominant. Consider a strategy profile in which low type of player A always play Aggressive and low type of player B always play Passive. The high value type of either player chooses the dominant action, Aggressive. Then this constitutes a BNE. Another possibility is a partially mixed strategy in which the low type of both player mix with the same probability that depends on β.

7.6 Evolutionary Equilibrium

Solution 7.6.1.

(A) All three payoff combinations along the main diagonal represent strong Nash equilibrium (NE) points, so the corresponding vertices in figure 7.8 are all EE's.

(B) Here the only NE is a 50:50 mixture of strategies a and b, suggesting a possible EE at the corresponding population distribution over these pure strategies. (Strategy c, since its payoffs are dominated by those of both a and b at all population proportions, becomes extinct in the EE.) Since each of a and b does better the larger the proportion of the other strategy in the population (compare payoff pattern II in figure 7.7), the population is "repelled" from the a and b vertices and does indeed evolve toward the 50:50 distribution along the a-b edge.

(C) The only NE is a $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$ mixture. As indicated by the low-payoff pairs along the main diagonal of the matrix, here each strategy does worst of all at its own vertex. So the population is repelled from all three vertices and evolves toward the corresponding interior distribution.

(D) Here there are weak pure-strategy NE's at all four upper-left cells of the matrix. Strategy c, being dominated throughout, must become extinct in the EE. But this having occurred, a and b have equal payoffs as do all mixtures of them as well. So there is no determinate evolutionary trend toward any particular point along the a-b edge, though the edge as a whole is an evolutionary equilibrium region.

Solution 7.6.2.

(A) Let the population distribution be (p_1, p_2, p_3). Then

$$V_a = 2p_1 + 0p_2 + 0p_3 = 2p_1$$
$$V_b = 0p_1 + p_2 + 0p_3 = p_2$$
$$V_c = 0p_1 + 2p_2 + p_3 = 2p_2 + p_3.$$

Plugging into the footnote gives the desired system.

(B) Sum up the system derived in (A) and set to zero to get

$$\kappa p_1(2p_1 - V) + \kappa p_2(p_2 - V) + \kappa p_3(2p_2 + p_3 - V) = 0$$

Cancel κ and rearrange to get

$$V = 2p_1^2 + (1 - p_1)^2.$$

(C) $\Delta p_1 > 0$ if and only if $2p_1 - V > 0$, if and only if(by (B))

$$2p_1 - 2p_1^2 - (1 - p_1)^2 > 0,$$

and one can verify the above inequality holds if and only if $1/3 < p_1 < 1$.

(D) Solve

$$\min 2p_1^2 + (1 - p_1)^2 = 3p_1^2 - 2p_1 + 1$$

to get $6p_1 = 2$. Plugging in $p_1 = 1/3$ into the objective function, we can obtain the minimum which is $2/3$.

Now, if $p_2 < 2/3$, then $p_2 - V < 0$, hence $\Delta p_2 > 0$.

(E) If $p_1 < 1/3$ and $p_3 \approx 0$, then $p_1 + p_3 < 1/3$, so $p_2 > 2/3$. Furthermore, $p_2 \approx 1 - p_1 > 2p_1^2 + (1 - p_1)^2$, so $p_2 - V > 0$, which implies $\Delta p_2 > 0$.

(F) Let G denote type 1 and O denote type 2. (C) implies that if $p_1 > 1/3$ then type 1 will grow in the population. (E) implies that if the population distribution is in the lower left corner type 2 will grow.

Solution 7.6.3.

(A) If $p_c = 1 - \epsilon$, and $p_b > p_a$, then

$$V(b|p) = 3\epsilon + 2(1-\epsilon) > 4p_a + 2p_b + 2(1-\epsilon) = V(c|p),$$

where $V(x|p)$ is the expected utility of using strategy x when the population is distributed as p.

(B) Given the mixture $s = xa + (1-x)b$, $V(s) = xV(a|p) + (1-x)V(b|p)$ because expected utility is linear. Then we have

$$xV(a|p) + (1-x)V(b|p) - V(c|p) = 3\epsilon + x(1-\epsilon) + (1-x)2(1-\epsilon) - 4p_1 - 2p_2 - 2(1-\epsilon) > 0$$

if and only if

$$3\epsilon - 4p_1 - 2p_2 > 4x(1-\epsilon)$$

Hence for any x, we can find ϵ large enough such that the above inequality holds, given $p_1 < p_2$. It then shows that the dynamic on the abc triangle around the ab-edge(where ϵ is large) will be pushed away from c. Along the ab edge, where $\epsilon = 1$, a and b has no difference, thus every point will remain fixed. However, in the neighborhood, strategy b has a slight advantage over a when their users encounter strategy c, so population will move toward b.

Solution 7.6.4.

(A) There are two pure strategy NEs: $(x_1^j, x_1^k), (x_2^j, x_2^k)$, and a mixed strategy NE in which player i, j play x_1^i, x_1^j with probability 2/5 respectively.

(B) The EEs are $(1,0;1,0)$ and $(0,1;0,1)$. Since there are only 2 dimensions, we can represent the possible states in the $[0,1] \times [0,1]$ box, where the first EE, $p = q = 1$, is in the upper right corner. The reason is as follows. If $q_1 > 2/5$, then for the row population type 1 agents will be more advantageous, since they can coordinate better with the opposing population. Similarly, if $q_1 < 2/5$, then it's more advantageous to be type 2 in the row population. Run the same reasoning for the column population, we see that when $p_1 > 2/5, q_1 > 2/5$, the population will grow toward $(1,0;1,0)$.

When p_1 is large and q_1 is small, the population will either grow toward the $(1,0;1,0)$ corner or the $(0,1;0,1)$ corner, or the mixed strategy NE distribution $(2/5, 3/5; 2/5, 3/5)$. But this point is not EE since points around the 45-degree line that crosses it will move away from it.

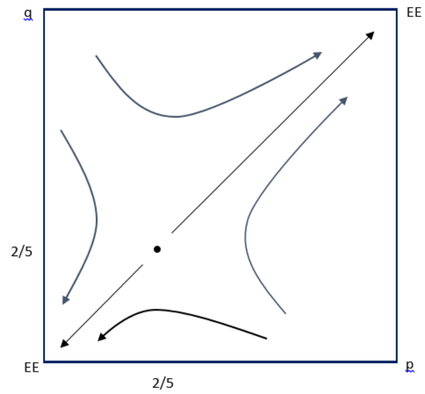

Figure 4: Ex 7.6.4(B)

(C) The result will differ. Intuitively, suppose the row player are all coward and the column all hero. Then it is unwise for the row type to play hero and the column to play coward. The situation is anti coordination, contrary to what we have in (B). The EEs will now be $(1,0;0,1)$ and $(0,1;1,0)$. The mixed strategy NE will not be an EE distribution because in it's neighborhoods along the diagonal line, the distribution will be drawn toward the two corner EEs just mentioned.

Solution 7.6.5.

(A) First note that positive probability mass at some point $\bar{t}$ means discontinuity of the distribution function at $\bar{t}$. Suppose player 2 plays the distribution $G(t)$ which is discontinuous at $\bar{t}$. Then for player 1, the expected payoff for playing t is given by

$$E[U_1(t,G)] = \int_0^t (V - cx)dG(x) - ct(1 - G(t)). \tag{1}$$

Since $G(t)$ is discontinuous, the expected utility jumps down at $\bar{t}$. Let $H(t)$ be the best response to $G(t)$. Then on $[\bar{t}, \bar{t}+\epsilon)$, H will put zero probability. But a better response to such H for player 2 will be then decrease the probability on $\bar{t}$, since his opponent will not play anything in $[\bar{t}, t+\epsilon)$. Hence, any pair of distribution with positive probability mass will not be mutual best responses.

(B) This is just equation (1).

(C) Suppose $G(t)$ is a symmetric NE. Then for players to mix, they must be indifferent between any pure strategies in the support of $G(t)$. Hence $E[U_i(t,G)]$ is a constant. Differentiate (1) w.r.t. t to obtain

$$(V - ct)G'(t) - c + ctG'(t) + cG(t) = 0$$

One can then check

$$G(t) = 1 - e^{-ct/V}$$

is the solution satisfying the boundary condition $G(\infty) = 1$.

(D) Let $c/V = k$. Then the equilibrium is of the form $G(t) = 1 - e^{-kt}$. Hence it reduces to choose the unit of the time. We can therefore, for simplicity, choose $V = c$.

(E) Without loss of generality take $V = c = 1$. We will use a series of integration by parts, and using the technique $v' = -d(1-v)$ each time. Let μ be the distribution $H(t)$ and ν be the distribution $G(t)$. By (1), the expected utility is

$$\begin{aligned} V(\nu|\mu) &= \int_0^\infty \left(\int_0^t (1-x)H'(x)dx - t(1-H(t)) \right) G'(t)dt \\ &= \int_0^\infty (1-t)(1-G(t))H'(t)dt - \int_0^\infty \left(-t(1-G(t)) + \int_0^t (1-G(x))dx \right) H'(t)dt \\ &= \int_0^\infty (1-G(t))H'(t)dt + \left(\int_0^t 1-G(x)dx \right) (1-H_(t)) \Big|_0^\infty - \int_0^\infty (1-G(t))(1-H(t))dt \\ &= \int_0^\infty (1-G(t))(H'(t)-1+H(t))dt. \end{aligned}$$

(F) By (E),

$$V(\mu|\mu) - V(\nu|\mu) = \int_0^\infty (1-H(t))(H'(t)-1+H(t))dt - \int_0^\infty (1-G(t))(H'(t)-1+H(t))dt.$$

Now we solve

$$\max_\mu V(\mu|\mu) - V(\nu|\mu).$$

The Euler equation w.r.t. H and H' is

$$-H' - (1-G) + \frac{d}{dt}(1-H-(1-G)) = 0,$$

which is zero when plugging in $H = G$ and using $G' + G - 1 = 0$. Hence

$$V(\mu|\mu) - V(\nu|\mu) < V(\nu|\nu) - V(\nu|\nu) = 0.$$

This shows that strategy ν is protected from the invasion of strategy μ for all μ, i.e., ν is an EE.