The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 6: Information and Markets

6.1 The inter-related equilibria of prior and posterior markets

Solution 6.1.1. The condition for equilibrium for an non-informative situation is

$$\frac{j_{sm}\partial v(c)/\partial c_{smg}}{j_{s'm'}\partial v(c)/\partial c_{s'm'g'}} = \frac{P_{smg}}{P_{s'm'g'}}.$$

For informative equilibrium, we first write down the agent's utility maximization problem for the sake of completeness. Let P^I_{smg} denote the posterior price after receiving message m, and P^0_{smg} denote the prior price. The agent solves

$$\max_{c_{smg},\tilde{c}_{smg}} \sum_m q_m \sum_s \pi_{s\cdot m} v(c_{smg})$$

s.t.

$$\begin{aligned} &\sum_{sg} P^I_{smg} c_{smg} = \sum_{sg} P^I_{smg}\tilde{c}_{smg}, \qquad \forall m \\ &\sum_{smg} P^0_{smg}\tilde{c}_{smg} = W. \end{aligned}$$

The agent optimizes in two steps. First, he trades the contingent bundle $\tilde{c}$, and later, after he receives message m, he trades c_{smg} over all s, g, with the budget being the contingent wealth of $\tilde{c}$ given m. Let λ_m and μ be the Lagrange multipliers of the first and second constraints, respectively. The first-order condition is given by

$$\begin{aligned} q_m \pi_{s\cdot m} v'(c_{smg}) &= \lambda_m P^I_{smg} \\ \lambda_m P^I_{smg} &= \mu P^0_{smg}. \end{aligned}$$

Hence, given s, m, we have

$$\frac{v'(c_{smg})}{v'(c_{smg'})} = \frac{P^I_{smg}}{P^I_{smg'}} = \frac{P^0_{smg}}{P^0_{smg'}},$$

and given s, g, we have

$$\frac{j_{sm}v'(c_{smg})}{j_{sm'}v'(c_{sm'g})} = \frac{\lambda_m}{\lambda_{m'}}\frac{P^I_{smg}}{P^I_{sm'g}} = \frac{P^0_{smg}}{P^0_{sm'g}}.$$

Since the allocation (c_{smg}) is already optimal under the prices (P^0_{smg}) before and after one receives the message m, no posterior trade is needed. Proposition 1 still holds.

Solution 6.1.2. If the information is revealed prematurely, there will be no chance to trade and individuals of one type are very adversely affected. In particular, each agent's expected utility is $-\infty$ as

$$EU_i = 0.6 \ln 400 + 0.4 \ln 0 = -\infty.$$

Similar for agent j. Hence their willingness to pay for the information not to be revealed is infinite.

Solution 6.1.3. First we calculate the CCM equilibrium. Note that for $v(n, f) = \ln n + \ln f$, the demand is given by $c_g = \pi_s (W/2)/P_{gs}$, where $g \in \{n, f\}$. Since both agents have the same endowment, their wealth is the same. The market clearing condition is

$$\begin{aligned}
0.5\frac{W/2}{P_{n1}} + 0.6\frac{W/2}{P_{n1}} &= 200 \\
0.5\frac{W/2}{P_{n2}} + 0.4\frac{W/2}{P_{n2}} &= 200 \\
0.5\frac{W/2}{P_{n1}} + 0.6\frac{W/2}{P_{n1}} &= 400 \\
0.5\frac{W/2}{P_{n2}} + 0.4\frac{W/2}{P_{n2}} &= 160
\end{aligned}$$

Using the normalization $P_{n1} + P_{n2} = 1$, we get

$$(P_{n1}, P_{n2}, P_{f1}, P_{f2}) = (\frac{11}{20}, \frac{9}{20}, \frac{11}{40}, \frac{9}{16}).$$

Under such prices the wealth of each agent is $W = 200$. To attain the CCM allocation using an NCM regime, note that in CCM the contingent wealth implied by the optimal allocation is $W^s = \pi_s W$. We now show that under NCM with the same prior and posterior price, the agents can achieve the same contingent wealth. Let $\tilde{c}^i_{ns}$ be agent i's prior round trade of good n. For agent 1, he has $W^s = 100$ for all s. So we need

$$200 \times \frac{11}{40} + \tilde{c}^1_{n1} \times \frac{11}{20} = 100$$

and

$$80 \times \frac{9}{16} + \tilde{c}^1_{n2} \times \frac{9}{20} = 100.$$

Thus we obtain

$$\tilde{c}^1_{n1} = \frac{900}{11}, \quad \tilde{c}^1_{n2} = \frac{1100}{9}.$$

A similar argument for agent 2 implies

$$\tilde{c}^2_{n2} = \frac{1300}{11}, \quad \tilde{c}^2_{n2} = \frac{700}{9}.$$

Note that the prior round market clears under the given prices and allocations. Hence with the above prior round trade, the agents will be able to attain the CCM equilibrium allocation.

6.2 Speculation and future trading

Solution 6.2.1.

(A) Since under $P = (0.6, 0.4, 0.3, 0.5)$ each agent's wealth is still 200, the optimal allocation under the CCM regime will remain the same.

(B) It suffices to show that after the prior round trade the agent's contingent wealth is $(W^1, W^2) = (120, 80)$. Agent 1's contingent allocation is now

$$C = (\frac{800}{3}, 150 + \frac{800}{3}, 200 - \frac{1000}{3}, 160 - \frac{1000}{3}).$$

If $s = 1$, then his wealth is

$$\frac{800}{3} \times 0.6 + (200 - \frac{1000}{3}) \times 0.3 = 120.$$

Similarly, if $s = 2$, his wealth is

$$\frac{1250}{3} \times 0.4 - \frac{520}{3} \times 0.5 = 80.$$

(C) His state contingent wealth is exactly the same as that induced by the CCM allocation in the posterior round, hence with the same price vector he can obtain the optimal allocation.

(D) He can use his wealth in the posterior round to buy goods to settle futures contracts.

Solution 6.2.2. The martingale property in general does not hold. We will formally set up the agent's utility maximization problem and derive the equilibrium condition for prices. Let g_n, g_f denote the future trades of n, f in the prior round. The consumer's maximization problem is given by

$$\max_{c_{ns}, c_{fs}, g_n, g_f} \sum_s \pi_s v(c_{ns}, c_{fs})$$

subject to

$$P_n^0 g_n + P_f^0 g_f = 0 \tag{1}$$

$$\forall s: \quad P_{ns}^I c_{ns} + P_{fs}^I c_{fs} = P_{ns}^I (\bar{c}_{ns} + g_n) + P_{fs}^I (\bar{c}_{fs} + g_f) \tag{2}$$

Let μ be the multiplier of (1) and λ_s be the multipliers of (2) for each s. The first-order condition is given by

$$
\begin{aligned}
c_{ns}: &\quad \pi_s \frac{\partial v(c_{ns}, c_{fs})}{\partial c_{ns}} = \lambda_s P^I_{ns} \\
c_{fs}: &\quad \pi_s \frac{\partial v(c_{ns}, c_{fs})}{\partial c_{fs}} = \lambda_s P^I_{fs} \\
g_n: &\quad \mu P^0_n + \sum_s \lambda_s P^I_{ns} = 0 \\
g_f: &\quad \mu P^0_f + \sum_s \lambda_s P^I_{fs} = 0
\end{aligned}
$$

Hence,

$$
\frac{P^0_f}{P^0_n} = \frac{\sum_s \lambda_s P^I_{fs}}{\sum_s \lambda_s P^I_{ns}} = \frac{E[\partial v/\partial c_{fs}]}{E[\partial v/\partial c_{ns}]},
$$

which shows that the prior-round price ratio of the goods equals the ratio of expected marginal utilities of the goods.

If the numerator and denominator of the given equation is changed, one can check that the martingale property will also not hold. In general, let $E[X/Y] = a/b$ and $f(z) = 1/z$. Jensen's inequality implies $E[Y/X] = E[f(X/Y)] \geq f(E[X/Y]) = b/a$.

6.3 The production and dissemination of information

6.3.1 Private information and the leakage problem

Solution 6.3.1.1.

(A) Because of log utility, the demand is

$$
c^\omega_1 = \frac{\pi^\omega W}{P_1} = \frac{\pi^\omega (P_1 \overline{c^\omega_1} + P_2 \overline{c^\omega_2})}{P_1}. \tag{3}
$$

(B) Note that the market clearing is given by

$$
C^I_1 + C^U_1 = \overline{C^I_1} + \overline{C^U_1}.
$$

Plugging in (3) to obtain

$$
\pi^I (\overline{C^I_1} + \frac{P_2}{P_1} \overline{C^I_2}) + \pi^U (\overline{C^U_1} + \frac{P_2}{P_1} \overline{C^U_2}) = \overline{C^I_1} + \overline{C^U_1}.
$$

Rearrange to obtain

$$
\frac{P_2}{P_1} = \frac{(1-\pi^U)\overline{C^U_1} + (1-\pi^I)\overline{C^I_1}}{\pi^U \overline{C^U_2} + \pi^I \overline{C^I_2}}. \tag{4}
$$

(C) Given P_2/P_1, one can obtain π^I from (4).

(D) In this case, the price ratio becomes

$$\frac{P_2}{P_1} = \frac{(1-\pi^U)\overline{C_1^U} + (1-\pi^I)(\overline{C_1^I} - \delta_1)}{\pi^U \overline{C_2^U} + \pi^I(\overline{C_2^I} - \delta_2)}.$$

But then the uninformed cannot infer π^I from P_2/P_1, unless δ_1, δ_2 are known.

Solution 6.3.1.2.

(A) We have

$$P_1^A = P_1 + P_2$$

$$P_2^A = z_1 P_1 + z_2 P_2.$$

Hence

$$\frac{P_1^A}{P_2^A} = \frac{z_1 P_1 + z_2 P_2}{P_1 + P_2} = \frac{z_1 + z_2 \frac{P_2}{P_1}}{1 + \frac{P_2}{P_1}},$$

where P_2/P_1 is given by (4).

(B) By observing P_1^A/P_2^A, one can first infer P_2/P_1 from the above equation, and then obtain π^I from (4).

Solution 6.3.1.3.

(A) Similar to Ex 6.3.1.1(A), the price ratio is given by

$$\frac{P_2}{P_1} = \frac{(1-\pi^U)\overline{C_1^U} + (1-\pi^A)\overline{C_1^A} + (1-\pi^B)\overline{C_1^B}}{\pi^U \overline{C_2^U} + \pi^A \overline{C_2^A} + \pi^B \overline{C_2^B}}.$$

(B) Let

$$f(\pi^A, \pi^B) = \frac{(1-\pi^U)\overline{C_1^U} + (1-\pi^A)\overline{C_1^A} + (1-\pi^B)\overline{C_1^B}}{\pi^U \overline{C_2^U} + \pi^A \overline{C_2^A} + \pi^B \overline{C_2^B}}.$$

The question amounts to whether, generically, f is a one-to-one function given that its domain (the set of possible (π^A, π^B)) is finite. It should be clear that if one randomly picks finitely many points in $S = [0,1] \times [0,1]$, then, with probability one, f will be one-to-one. The above claim holds for arbitrary C^I, C^A, C^B. For example, consider the case $C^A = C^B = C^I$, then the value of f (the price ratio) is determined solely by $\pi^A + \pi^B$. So we can partition S into $\cup_{k \in [0,1]} S_k$ where $S_k = \{(\pi^A, \pi^B) \in S | \pi^A + \pi^B = k\}$. Obviously, the probability of picking two points in the same S_k is zero, since the index k is a continuum.

(C) If one randomly picks a continuum of points, for example circle an area in S, one will with positive probability circle points (π^A, π^B) that produce the same price ratio.

6.3.2 Partial leakage with constant absolute risk aversion

Solution 6.3.2.1.

(A) Let $\Sigma_{nn} = [cov(R+\epsilon_i, R+\epsilon_j)]$, $\Sigma_{1n} = (cov(R, R+\epsilon_1), ..., cov(R, R+\epsilon_n)) = (\sigma^2, ..., \sigma^2)$ and $\mathbf{m} - \mu$ a vector of length n with entries $m_i - \mu$. A standard result for multivariate normal random variables implies that

$$\mathrm{E}[R|m_1, ..., m_n] = \mu + \Sigma_{1n}\Sigma_{nn}^{-1}(\mathbf{m} - \mu).$$

Observe that Σ_{nn} is a matrix with diagonal entries $\sigma^2 + \sigma_\epsilon^2$ and off-diagonal entries σ^2. By symmetry, the inverse of Σ_{nn} will also be of this form. This implies that $\Sigma_{1n}\Sigma_{nn}^{-1}$ will be a vector with all entries the same, say, β. Then

$$E[R|m_1, ..., m_n] = \mu + \beta(\sum_i m_i - n\mu) = (1 - \beta n)\mu + \beta n \frac{m_1 + ... + m_n}{n}.$$

(B) Suppose there are n insiders and m outsiders. Then it follows from (6.3.5) that

$$\sum_{i=1}^{n} \frac{\mathrm{E}[R|m_i] - P}{A_i \operatorname{Var}(R|m_i)} + \sum_{i=1}^{m} \frac{\mu - P}{A_i \sigma^2} = Q_2, \tag{5}$$

where

$$\begin{aligned} \mathrm{E}[R|m_i] &= \frac{\sigma_\epsilon^2}{\sigma^2 + \sigma_\epsilon^2}\mu + \frac{\sigma^2}{\sigma^2 + \sigma_\epsilon^2} m_i \\ \operatorname{Var}[R|m_i] &= \frac{\sigma^2\sigma_\epsilon^2}{\sigma^2 + \sigma_\epsilon^2}. \end{aligned}$$

Plugging into (5) and rearranging

$$P\left(\frac{n(\sigma^2 + \sigma_\epsilon^2)}{\sigma^2\sigma_\epsilon^2} + \frac{m}{A_i\sigma^2}\right) = \frac{n}{A_i\sigma^2}\frac{1}{A_i\sigma_\epsilon^2}\sum_{i=1}^{n} m_i + \frac{m\mu}{A_i\sigma^2} - Q_2.$$

Hence P is a function of $\sum_i m_i$.

If every insider receives $(m_1, ..., m_n)$, then equation (6.3.5) becomes

$$\sum_{i=1}^{n} \frac{\mathrm{E}[R|m_1, ..., m_n] - P}{A_i \operatorname{Var}(R|m_1, ..., m_n)} + \sum_{i=1}^{m} \frac{\mu - P}{A_i\sigma^2} = Q_2.$$

By (A), the equilibrium price P is still linear in $\sum m_i$,

(C) Suppose k out of n insiders purchase information. Then if one additional insider buys information he loses c units of riskless asset, but he can purchase the risky asset with $\mathrm{E}[R|m_1, ..., m_{k+1}], \mathrm{Var}[R|m_1, ..., m_{k+1}]$ under the equilibrium price $P(m_1, ..., m_{k+1})$. If the expected equilibrium price of the risky asset rises, then there will be a trade-off. Hence one may guess that there is an optimal k.

(D) Insiders now have different incentives to purchase information. If differences in risk aversion are small, then the answer will not differ much from that in (C).