The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 3: Comparative Statics of the Risk-Bearing Optimum

3.1 Measures of Risk Aversion

Solution 3.1.1.

(A) The quadratic utility function has the form

$$v(c) = k_0 + k_1 c - \frac{1}{2} k_2 c^2$$

where $k_1, k_2 > 0$. Furthermore, we restrict the domain of c to be $0 \leq c < k_1/k_2$ so the utility function is strictly increasing in this region, and is increasing absolute risk aversion and increasing relative risk aversion. Hence

$$A(c) = \frac{k_2}{k_1 - k_2 c}$$

is increasing for every $c < k_1/k_2$. Further

$$R(c) = \frac{k_2 c}{k_1 - k_2 c}$$

is also increasing for every $c < k_1/k_2$.

(B) We have

$$A(c) = \frac{1}{\alpha + \beta c}$$
$$R(c) = \frac{c}{\alpha + \beta c}.$$

So

$$A'(c) = \frac{\beta}{(\alpha + \beta c)^2} \leq 0$$

if and only if $\beta \leq 0$.

$$R'(c) = \frac{\alpha}{(\alpha + \beta c)^2} \geq 0$$

if and only if $\alpha \geq 0$.

Solution 3.1.2.

(A) If $v(c) = -e^{-Ac}$, then $v'(c) = Ae^{-Ac}$, $v''(c) = -A^2e^{-Ac}$, hence $A(c) = -v''/v' = A$. To show uniqueness, we solve the ordinary differential equation

$$-\frac{v''}{v'} = A.$$

Rewrite the equation as

$$\frac{d\ln v'(c)}{dc} = -A.$$

Integrate both sides to get

$$\ln v'(c) = -Ac + B.$$

Take exponential on both sides and integrate again to get

$$v(c) = \frac{1}{-A}e^{-Ac+B} + D = -\frac{e^B}{A}e^{-Ac} + D.$$

Here B and D are arbitrary constants.

(B) This follows from direct computations. Note that when $R > 1$, the utility function will be $v(c) = -c^{1-R}$ exhibits CRRA, so that $v' > 0$.

Solution 3.1.3.

(A) Yes. For example, consider $G = (100, -1/0.99, 0.01, 0.99)$. Then $E[G] = 0, E[(G - E[G])^3] = 100^3 \times 0.01 + (-1/0.99)^3 \times 0.99 >> 0$.

(B) The Taylor expansion of $v(c)$ around the mean $\mu = E[\tilde{c}]$ is given by

$$v(c) \approx v(\mu) + v'(\mu)(c-\mu) + \frac{v''(\mu)(c-\mu)^2}{2} + \frac{v'''(\mu)(c-\mu)^3}{6}. \tag{1}$$

Substituting $\tilde{c}$ for c in (1) and taking expectation yields

$$E[v(c)] \approx v(\mu) + \frac{v''(\mu)E[(\tilde{c}-\mu)^2]}{2} + \frac{v'''(\mu)E[(\tilde{c}-\mu)^3]}{3!}.$$

Hence $v''' > 0$ implies positive-skewness preference.

(C)

$$\frac{d}{dc}\left(-\frac{v''(c)}{v'(c)}\right) = \frac{-v'''(c)v'(c) + (v''(c))^2}{(v'(c))^2} < 0$$

implies

$$v'''(c) > \frac{(v''(c))^2}{v'(c)} > 0.$$

Solution 3.1.4. Let $v_J(c) = f(v_K(c))$ where $f' > 0$. Then

$$A_J(c) = -\frac{f'(v_K(c))v_K''(c) + f''(v_K(c))v_K'^2(c)}{f'(v_K(c))v_K'(c)}$$
$$= A_K(c) - \frac{f''(v_K(c))v_K'(c)}{f'(v_K(c))}.$$

Hence $A_J(c) > A_K(c)$ if and only if $f'' < 0$. This shows (A) and (B).

Solution 3.1.5.

(A) Without loss of generality, assume $v(c) = -e^{-Ac}$. The agent rejects the small gamble if

$$-\frac{1}{2}e^{-110A} - \frac{1}{2}e^{100A} < -1,$$

which is equivalent to

$$e^{-110A} + e^{100A} > 2.$$

Using numerical methods (the quickest way is an Excel spreadsheet), one can find the smallest A for which the above inequality holds is $A = 0.001$.

(B) No. If $A \geq 0.001$, then $e^{1000A} \geq e^{10} > 2$, hence the agent will surely reject the second gamble as

$$-\frac{1}{2}e^{-GA} - \frac{1}{2}e^{1000A} < -1,$$

for any G.

Solution 3.1.6.

(A) A 2nd order Taylor series expansion of v around $\mu = E[\tilde{c}]$ gives

$$v(c) \approx v(\mu) + v'(\mu)(c-\mu) + \frac{v''(\mu)}{2}(c-\mu)^2.$$

Hence

$$v(\mu - b) = E[v(\tilde{c})] \approx v(\mu) + \frac{v''(\mu)}{2}\sigma^2.$$

By the mean value theorem, there exists $\hat{c} \in (\mu - b, \mu)$ such that

$$v'(\hat{c})b \approx -\frac{v''(\mu)}{2}\sigma^2.$$

Hence

$$b \approx -\frac{v''(\mu)}{v'(\hat{c})}\frac{\sigma^2}{2}.$$

(B) When the risk is small, we can apply (A) to the utility function given by $v(w+c)$, and then DARA implies

$$b_1 \approx -\frac{v''(\mu+w)}{v'(c''+w)}\frac{\sigma}{2} < -\frac{v''(\mu)}{v'(c')}\frac{\sigma^2}{2} \approx b_0.$$

(C) Let $\overline{v} = v(c+w)$. Suppose

$$\frac{-v''(c+w)}{v'(c+w)} < \frac{-v''(c)}{v'(c)}.$$

Then by Exercise 3.1.4, there exists an increasing and convex function f such that $v(c+w) = f(v(c))$. Now by the definition of b_1 and Jensen's inequality,

$$f(v(\mu - b_1)) = v(\mu + w - b_1) = E[v(w+\tilde{c})] = E[f(v(\tilde{c})] > f(E[v(\tilde{c}]) = f(v(\mu - b_0)).$$

Hence $b_1 < b_0$.

Solution 3.1.7.

(A) Equation (i) gives

$$\frac{1}{3}v(\overline{c}) + \frac{1}{3}v(\overline{c}+e) + \frac{1}{3}v(\overline{c}-e) = v(\overline{c}-b_0).$$

Equation (iii) gives

$$\frac{1}{3}v(w+\overline{c}) + \frac{1}{3}v(\overline{c}+e) + \frac{1}{3}v(\overline{c}-e) = \frac{1}{3}v(w+\overline{c}-b_2) + \frac{2}{3}v(\overline{c}-b_2).$$

Substitute (i) into (ii) and rearrange to get

$$[v(\overline{c}+w) - v(\overline{c}+w-b_2)] - [v(\overline{c}) - v(\overline{c}-b_2)] = 3[v(\overline{c}-b_2) - v(\overline{c}-b_0)]. \tag{2}$$

(B) As v is concave, the left-hand side of (2) is negative. Hence, $b_2 > b_0$.

(C) Observe from (A) that for fixed b_2, a rise in w makes the left-hand side more negative. Hence, for the equation to hold, b_2 must also rise. Thus, the larger the w (the larger income uncertainty is), the larger b_2 the agent is willing to pay. This result is solely due to the fact that $v'' < 0$. That is, if the person is risk averse, adding more risk makes his risk premium larger.

3.2 Endowment and Price Effects

3.2.1 Complete Markets

Solution 3.2.1.1.

(A) The Fundamental Theorem of Risk Bearing implies

$$\frac{\pi_s v'(c_s)}{p_s} = \lambda \quad \forall s. \tag{3}$$

Suppose c_s rises when p_s rises, then $\pi_s v'(c_s)/p_s$ decreases. As $p_{s'}$ stays the same for all $s' \neq s$, it implies $c_{s'}$ rises for all $s' \neq s$. But this violates the budget constraint.

(B) Part (A) implies a rise in p_s will result to a decrease in holdings of c_s. For a net seller it implies he will sell more of c_s.

Solution 3.2.1.2.

(A) Let $c_s(p_1, ..., p_S) = c_s(p)$ be the demand function for state-s claim. Assume own price elasticity $e < -1$, that is,

$$\frac{\partial c_s(p)}{\partial p_s} \frac{p_s}{c_s(p)} < -1.$$

Then

$$\frac{\partial p_s c_s(p)}{\partial p_s} = c_s(p) + \frac{p_s \partial c_s(p)}{\partial p_s} < c_s(p) - c_s(p) = 0.$$

Hence the spending on state-s claim decreases. As a result, spending on state-s' claims will increase (at least for some s'). Equation (3) then implies for all s' the state-s' claims increase. Hence $\partial c_{s'}(p)/\partial p_s > 0$.

(B) No, the own price elasticity must be larger than -1, otherwise it contradicts what's proved in (A), as shown by the graph.

(C) Let $s' \neq s$. Equation (3) implies

$$\frac{\pi_s v'(c_s)}{p_s} = \frac{\pi_{s'} v'(c_{s'})}{p_{s'}}.$$

Hence

$$\pi_s c_s v'(c_s) = \frac{p_s c_s \pi_{s'} v'(c_{s'})}{p_{s'}}.$$

Differentiate both sides with respect to p_s to obtain

$$\pi_s \left(\frac{\partial c_s}{\partial p_s} v'(c_s) + c_s v''(c_s) \frac{\partial c_s}{\partial p_s} \right) = \frac{\pi_{s'}}{p_{s'}} \left(c_s v'(c_{s'}) + p_s v'(c_{s'}) \frac{\partial c_s}{\partial p_s} + p_s c_s v''(c_{s'}) \frac{\partial c_{s'}}{\partial p_s} \right). \tag{4}$$

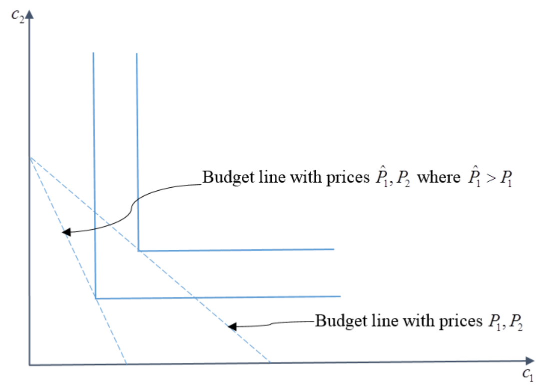

Figure 1: Ex 3.2.1.2(B)

Suppose that own price elasticity < -1. Then the first term in the right-hand side of (4) is negative. Furthermore, (A) will imply the second term in the right-hand side is also negative. It follows that the left-hand side must be negative. Rearranging and noting that $\partial c_s/\partial p_s < 0$, we get

$$1 + \frac{v''(c_s)c_s}{v'(c_s)} > 0,$$

which says relative risk aversion is less than 1.

Conversely, if the relative risk aversion is less than 1, then the left-hand side of (4) is negative. Suppose the own price elasticity is larger than -1, then the argument in (A) will show that $\partial c_{s'}/\partial p_s < 0$. Hence the two terms in the right-hand side of (4) will all be positive, which is a contradiction.

3.2.2 Incomplete Markets

Solution 3.2.2.1.

(A) As

$$v((\lambda q^\alpha + (1-\lambda)q^\beta) \cdot z_s) = v(\lambda q^\alpha \cdot z_s + (1-\lambda)q^\beta \cdot z_s)$$

and v is strictly concave, we have

$$v(\lambda q^\alpha \cdot z_s + (1-\lambda)q^\beta \cdot z_s) > \lambda v(q^\alpha \cdot z_s) + (1-\lambda)v(q^\beta \cdot z_s).$$

This holds for each vector z_s. It follows that for all s:

$$\pi_s v((\lambda q^\alpha + (1-\lambda)q^\beta) \cdot z_s) > \lambda \pi_s v(q^\alpha \cdot z_s) + (1-\lambda)\pi_s v(q^\beta \cdot z_s).$$

The concavity of $U(q)$ then follows by directly summing over s.

(B) Suppose q^α and q^β are on the same indifference curve which gives utility $\overline{U}$. Since U is strictly concave, for any convex combination of q^α and q^β, we have

$$U(\lambda q^\alpha + (1-\lambda)q^\beta) > \lambda U(q^\alpha) + (1-\lambda)U(q^\beta) = \overline{U}.$$

Hence the convex combination gives higher utility, which means the preference over q is convex.

Solution 3.2.2.2.

(A) The agent solves

$$\max_{q_1,\ldots,q_M} k_1 - k_2 E[e^{-A(\sum_{m=1}^M q_m z_m)}]$$

s.t.

$$\sum_{m=1}^M p_m q_m = \overline{W}.$$

Substituting $q_1 = (\overline{W} - \sum_{m=2}^M q_m p_m)/p_1$ and letting $z_1 = 1$, we get

$$\max_{q_2,\ldots,q_M} k_1 - k_2 e^{-A\frac{\overline{W}}{p_1}} E\left[e^{-A(-\sum_{m=2}^M \frac{p_m q_m}{p_1} + \sum_{m=2}^M q_m z_m)}\right].$$

(B) The first-order condition is then

$$\frac{\partial}{\partial q_m} E\left[e^{-A(-\sum_{m=2}^M \frac{p_m q_m}{p_1} + \sum_{m=2}^M q_m z_m)}\right] = 0$$

for $m = 2, ..., M$, which is independent of $\overline{W}$.

Solution 3.2.2.3. Let $k_2 = p_2 q_2/\overline{W}$. Then the budget constraint is $p_1 q_1/\overline{W} + k_2 = 1$. The utility is then

$$U(k_2) = E[v(q_1 z_1 + q_2 z_2)] = E[v((1-k_2)\frac{\overline{W}}{p_1})z_1 + \frac{\overline{W}k_2}{p_2} z_2].$$

Let

$$c = \frac{\overline{W}}{p_1} z_1 + \overline{W} k_2 \left(\frac{z_2}{p_2} - \frac{z_1}{p_1}\right) = (1+R_1)\overline{W} + \overline{W} k_2 (\tilde{R}_2 - R_1).$$

Then

$$U(k_2) = E[v((1+R_1)\overline{W} + \overline{W} k_2 (\tilde{R}_2 - R_1))].$$

Hence the optimal k_2^* satisfies the FOC

$$U'(k_2^*) = E[\overline{W}(\tilde{R}_2 - R_1) v'(c^*)] = 0. \tag{5}$$

Now differentiate (5) with respect to $\overline{W}$ to get

$$\frac{d}{d\overline{W}}U'(k_2^*) = E[(\tilde{R}_2 - R_1)v'(c^*)] + E[\overline{W}(\tilde{R}_2 - R_1)v''(c^*)[(1+R_1) + k_2(\tilde{R}_2 - R_1)]]$$
$$= -E[(\tilde{R}_2 - R_1)v'(c^*)R(c^*)].$$

If $R(c)$ is constant, then $dU'(k_2^*)/d\overline{W} = 0$, which means k_2 is independent of $\overline{W}$. If $R(c)$ is increasing, then for all realizations of $\tilde{R}_2$ we have

$$(\tilde{R}_2 - R_1)R(c^*) > (\tilde{R}^2 - R^1)R((1+R_1)\overline{W}),$$

which implies an increase of $\overline{W}$ decreases the marginal utility of k_2, and thus the optimal k_2 will decrease. Similarly, if $R(c)$ is decreasing then the optimal k_2 increases with $\overline{W}$.

Solution 3.2.2.4.

(A) Substituting $v_J(c) = f(v_K(c))$ into (3.2.9) gives

$$U_J'(q_2^J) = P_2^A E[(\tilde{R}_2 - R_1)f'(v_K(\tilde{c}))v_K'(\tilde{c})] = 0. \tag{6}$$

(B) For each possible realization of $\tilde{R}_2$, since f is concave, f' is decreasing, and we then have

$$(R_2 - R_1)f'(v_K(c)) < (R_2 - R_1)f'(v_K((1+R_1)\overline{W})).$$

Substituting back into (6) gives

$$U_J'(q_2^J) = 0 < f'(v_K((1+R_1)\overline{W}))P_2^A E[(\tilde{R}_2 - R_1)v_K'(\tilde{c})] = f'(v_K((1+R_1)\overline{W}))U_K'(q_2^J).$$

Hence $q_2^K > q_2^J$.

3.3 Changes in the Distribution of Asset Payoffs

Solution 3.3.1. From (3.1.1), expected utility is

$$U(q_2) = \sum_{s=1}^{S} \pi_s v(\overline{W}z_1 + q_2(z_{2s} - z_1)).$$

Differentiating w.r.t. q_2, the optimal holding q_2^* of the risky asset satisfies:

$$U'(q_2^*) = \sum_{s=1}^{S} \pi_s (z_{2s} - z_1)v'(\overline{W}z_1 + q_2^*(z_{2s} - z_1)) = 0.$$

Suppose z_{21} rises to $\hat{z}_{21}$. Holding q_2^* constant, only the first term in the summation changes. Then $U'(q_2^*)$ rises with z_{21} if

$$\phi(z_{21}) \equiv (z_{21} - z_1)v'(\overline{W}z_1 + q_2(z_{21} - z_1))$$

is an increasing function of z_{21}. Differentiating w.r.t z_{21} we obtain:

$$\begin{aligned}\phi'(z_{21}) &= v'(\cdot) + q_2^*(z_{21} - z_1)v''(\cdot) \\ &= v'(\cdot)\left[1 - \frac{q_2^*(z_{21} - z_1)}{\overline{W}z_1 + q_2^*(z_{21} - z_1)}R(\cdot)\right],\end{aligned}$$

where R is the degree of relative risk aversion.

From the first expression for ϕ' it follows immediately that, if $z_{21} \leq z_1$, $\phi'(z_{21})$ is positive. From the second, if $z_{21} > z_1$ and $R \leq 1$, then the bracket is positive and so again $\phi'(z_{21})$ is positive. We have therefore established that ϕ rises with z_{21}. Thus, for any $\hat{z}_{21} > z_{21}$, $U(q_2)$ is strictly increasing at $q_2 = q_2^*$. Since, as may readily be confirmed, $U(q_2)$ is a strictly concave function of q_2, it follows immediately that the new optimum holding of the risky assets exceeds q_2^*.

Remark 1. One feature of this exercise and of the type of parametric change assumed in the text is that the shift in the payoff distribution of the risky asset results in a lower mean return to holding the asset. From this observation it might be conjectured that clearer results would hold for changes in the distribution of consequences that preserve the expected return on the risky asset. In particular, if the expected payoff $E[\tilde{z}_2]$ were held constant but the variance $\sigma^2(\tilde{z}_2)$ were to rise, it seems plausible that an individual would reduce his demand for the risky asset. However, as the following exercise indicates, even this conjecture is false. Indeed it is possible for the mean return to rise and for the variance to fall, and yet for the optimal holding of the risky asset to remain unchanged.

Solution 3.3.2. (A) and (B) are confirmed by direct computation. To establish (C), suppose that the individual spends a proportion x of his wealth on the risky asset. Then his final consumption in state s is

$$\begin{aligned}c_s(x) &= xz_{2s} + (1 - x)50 \\ &= 50 + (z_{2s} - 50)x.\end{aligned}$$

Since $v(c) = -e^{-Ac}$, expected utility becomes

$$U(x) = -\pi e^{-Ac_1(x)} - (1 - \pi)e^{-Ac_2(x)}.$$

Differentiating w.r.t. x and making use of the expression for $c_s(x)$ we obtain

$$\begin{aligned}\frac{dU}{dx} &= A\pi e^{-Ac_1(x)}(z_{21} - 50) + A(1 - \pi)e^{-Ac_2(x)}(z_{22} - 50) \\ &= Ae^{-Ac_2(x)}[\pi(z_{21} - 50)e^{A(c_2(x) - c_1(x))} + (1 - \pi)(z_{22} - 50)].\end{aligned}$$

From the table we know that, in each of the three cases, $c_2(x) - c_1(x) = 30$ at $x = 1/2$. Moreover, by assumption $e^{30A} = 4$. Then, for each of the three cases:

$$\left.\frac{dU}{dx}\right|_{x=1/2} = Ae^{-Ac_2(x)}[4\pi(z_{21} - 50) + (1 - \pi)(z_{22} - 50)].$$

It is then a straightforward matter to confirm that the term in brackets is zero for each of the three cases. That is, even though assets α and β have the same mean and β has a higher variance, the optimal holding of the risky asset is the same. Moreover asset γ has a higher mean and lower variance than asset β and yet the optimal holding of the risky asset is still the same.

3.4 Stochastic Dominance

3.4.1 Comparison of Different Consumption Prospects

Solution 3.4.1.1. Using integration by parts with $u = v(c), dv = [F'(c) - H'(c)]dc$, for all increasing function $v(c)$

$$\begin{aligned} E_F[v(c)] - E_H[v(c)] &= \int_\alpha^\beta v(c)[F'(c) - H'(c)]dc \\ &= v(c)(F(c) - H(c))|_\alpha^\beta - \int_\alpha^\beta v'(c)[F(c) - H(c)]dc \\ &= \int_\alpha^\beta v'(c)[H(c) - F(c)]dc \qquad (7) \\ &> 0, \end{aligned}$$

where the last inequality follows from the fact that $v'(c)[H(c) - F(c)] \geq 0$ for all c with strict inequality from some c.

Solution 3.4.1.2. Let $I(c) = \int_\alpha^c [H(x) - F(x)]dx$. Integrating (7) by parts using $u = v'(c)$ and $dv = dI(c)$, we obtain

$$E_F[v(c)] - E_H[v(c)] = v'(c)I(c)|_\alpha^\beta - \int_\alpha^\beta v''(c)I(c)dc. \qquad (8)$$

As $v'(c)$ is increasing, the first term of (8) is positive. Also $v''(c) < 0$ and $I(c) > 0$ imply that the second term of (8) is negative. Hence, $E_F[v(c)] - E_H[v(c)] > 0$.

Solution 3.4.1.3. Assuming H is a mean-preserving spread of F, using integration by parts we then have

$$I(\beta) = (H(x) - F(x))x|_\alpha^\beta - \int_\alpha^\beta x(H'(x) - F'(x))dx = 0.$$

Hence the first term of (8) is zero. It then follows from $v'' < 0$ that

$$E_F[v(c)] > E_H[v(c)].$$

Remark 2. Observe that if $v(c)$ is convex, then the inequality will be reversed and then the agent will prefer the mean-preserving spread G to F.

Solution 3.4.1.4.

(A) Suppose F and G are not ranked by FOSD. Then there exists $c_1 \neq c_2$ such that $F(c_1) < G(c_1)$ and $F(c_2) > G(c_2)$. Define

$$v_i(c) = \begin{cases} -1 & c < c_i \\ 0 & c \geq c_i \end{cases}$$

Then

$$E_F[v_1(c)] = -F(c_1) > -G(c_1) = E_G[v_1(c)]$$

and

$$E_F[v_2(c)] = -F(c_2) < -G(c_1) = E_G[v_2(c)].$$

Hence there exist two increasing functions that rank F and G differently.

(B) Suppose F and G are not ranked by SOSD. Then there exists $c_1 \neq c_2$ such that

$$\int_\alpha^{c_1} F(c)dc < \int_\alpha^{c_1} G(c)dc$$

and

$$\int_\alpha^{c_2} F(c)dc > \int_\alpha^{c_2} G(c)dc.$$

Define

$$v_i(c) = \begin{cases} c - c_i & c < c_i \\ 0 & c \geq c_i \end{cases}$$

Then integration by parts shows

$$\begin{aligned} E_F[v_1(c)] &= \int_\alpha^{c_1} (c - c_1)F'(c)dc \\ &= (c - c_1)F(c)|_\alpha^{c_1} - \int_\alpha^{c_1} F(c)dc \\ &> -\int_\alpha^{c_1} G(c)dc \\ &= E_G[v_1(c)]. \end{aligned}$$

Similarly, $E_F[v_2(c)] < E_G[v_2(c)]$. Hence there exist two concave functions that rank F and G differently.

3.4.2 Responding to Increased Risk

Solution 3.4.2.1.

(A) This follows from an application of Ranking Theorem II.

(B) If $\partial/\partial x$ is convex in θ, then the ranking will be reversed, by the remark to Exercise 3.4.1.3.

Solution 3.4.2.2.

(A) The argument is exactly the same as Exercise 3.1.3(C).

(B) Let x be the saving and $\tilde{I}_2$ be the second-period income with distribution F. Then the agent solves

$$\max_x v_0(I_0 - x) + E_F[v_1((1+r)x + \tilde{I}_1)].$$

The first-order condition is

$$\frac{\partial E_F[v(x, \tilde{I}_1)]}{\partial x} = -v_0'(I_0 - x_F^*) + (1+r)E_F[v_1'((1+r)x_F^* + \tilde{I}_1)] = 0.$$

Suppose $v''' > 0$, then ranking theorem II, applied to the second term above, implies

$$\left.\frac{\partial E_G[v(x, \tilde{I}_1)]}{\partial x}\right|_{x = x_F^*} > 0.$$

Hence $x_G^* > x_F^*$.

Solution 3.4.2.3.

(A)

$$\begin{aligned} \mu x \frac{v''(\lambda + \mu x)}{v'(\lambda + \mu x)} &= (\mu x + \lambda - \lambda)\frac{v''(\lambda + \mu x)}{v'(\lambda + \mu x)} \\ &= -R(\lambda + \mu x) + \lambda A(\lambda + \mu x). \end{aligned}$$

(B) Suppose asset i costs p_i. Then for each q_2 and realization of $\tilde{z}_2$

$$c(q_2, z_2) = \overline{W}\frac{z_1}{p_1} + p_2 q_2 \left(\frac{\tilde{z}_2}{p_2} - \frac{z_1}{p_1}\right).$$

Then

$$A := \frac{\partial v(c(q_2, z_2))}{\partial q_2} = v'(c(q_2, z_2))p_2 \left(\frac{z_2}{p_2} - \frac{z_1}{p_1}\right).$$

Hence

$$\frac{\partial A}{\partial z_2} = v'(c(q_2, z_2)) \left(1 + \frac{v''(c(q_2, z_2))}{v'(c(q_2, z_2))} q_2 p_2 \left(\frac{z_2}{p_2} - \frac{z_1}{p_1}\right)\right).$$

(A) then implies the above equation is positive and decreasing. We can then apply ranking theorem II to conclude that if the distribution of $\tilde{z}_2$ becomes less favorable, then the optimal x^* will decrease.

(C) Assuming DARA and IRRA, we will have $\partial A/\partial z_2$ is decreasing, but not necessarily positive. With mean-preserving spreads, we can apply ranking theorem III to conclude that when the distribution of $\tilde{z}_2$ changes to a less favorable one, the optimal x^* will decrease.

Solution 3.4.2.4.

(A) As $\phi(x, v) = v^{-1}(x, c)$, the inverse function theorem shows $\phi'(x, v) = 1/v'(x, c)$. Further, $\hat{F}(v(c)) = F(c)$ implies $\hat{F}'(v(c))dv = F'(c)dc$. Apply the change of variable $c = \phi(x, v)$ to the integral

$$U'_F(x) = \int_\alpha^\beta \frac{\partial v(x, c)}{\partial x} F'(c)dc,$$

we get

$$\begin{aligned} U'_F(x) &= \int_{v_\alpha}^{v_\beta} \frac{\partial}{\partial x} v(x, \phi(x, v))F'(c)\phi'(x, v)dv \\ &= \int_{v_\alpha}^{v_\beta} \frac{\partial}{\partial x} v(x, \phi(x, v))\hat{F}'(v(c))dv, \end{aligned}$$

where $\alpha = \phi(x, v_\alpha)$ and $\beta = \phi(x, v_\beta)$.

(B) Suppose $\hat{G}(v)$ is a mean preserving spread of $\hat{F}$ at x^*, then

$$\int_{v_\alpha}^{v_\beta} v(x^*, c)\hat{F}'(v(c))dv = \int_{v_\alpha}^{v_\beta} v(x^*, c)\hat{G}'(v(c))dv.$$

Hence

$$\int_\alpha^\beta v(x^*, c)F'(c)dc = \int_\alpha^\beta v(x^*, c)G'(c)dc.$$

So the corresponding F and G gives the same expected utility at x^*.

(C) This follows directly from the expression obtained in (A).

(D) Differentiate both sides by c to obtain

$$y'(v(x,c))\frac{\partial v(x,c)}{\partial c} = \frac{\partial v(x,c)}{\partial c \partial x}.$$

Assume v is smooth so we can change the order of the cross derivative. Then

$$\begin{aligned} y'(v(x,c)) &= \frac{\partial/\partial x(\partial v(x,c)/\partial c)}{\partial v(x,c)/\partial c} \\ &= \frac{\partial}{\partial x}\ln(\frac{\partial v(x,c)}{\partial c}). \end{aligned} \tag{9}$$

(E) Differentiate (9) by c again to obtain

$$y''(v(x,c))\frac{\partial v(x,c)}{\partial c} = \frac{\partial^2}{\partial c \partial x}\ln(\frac{\partial v(x,c)}{\partial c}).$$

Assume the right-hand side is negative, then since $\partial v(x,c)/\partial c > 0$, $y''(v) < 0$. Then the optimal response theorem II follows from the expression in (A) and ranking theorem III.

Solution 3.4.2.5.

(A) Risk neutral agent maximizes expected profits, hence he solves

$$\max_q E[pq - C(q)].$$

The F.O.C. is then

$$\overline{p}_n \equiv E[p] = C'(q^*).$$

Under q^*, free entry condition implies the expected profit equals the outside option, i.e.,

$$\overline{p}_n q^* - C(q^*) = w.$$

(B) Suppose to the contrary that $\overline{p}_a \leq \overline{p}_n$. Then,

$$\begin{aligned} E[v(\tilde{p}q_a - C(q_a)) &< v(\overline{p}_a q_a - C(q_a)) && \text{by Jensen's inequality,} \\ &< v(\overline{p}_n q_a - C(q_a)) && \text{as } \overline{p}_a \leq \overline{p}_n \\ &< v(\overline{p}_n q_n - C(q_n)) && \text{as } q_n \text{ maximizes expected profit at price } \overline{p}_n \\ &= v(w). \end{aligned}$$

If this is true the firm is strictly better off taking its outside option.

(C) Let $f(q) = E[v(pq - C(q)]$ and therefore $f'(q) = E[v'(pq - C(q)](p - C'(q))]$. Differentiate the integrand of $f'(q)$ with respect to p to get:

$$\begin{aligned}\frac{\partial}{\partial p} v'(pq - C(q))(p - C'(q)) &= v''(pq - C(q))q(p - C'(q)) + v'(pq - C(q)) \\ &= v'(pq - C(q)) \left(1 + [pq - qC'(q)]\frac{v''(pq - C(q))}{v'(pq - C(q))}\right) \\ &= v'(pq - C(q))\left(1 + \phi(q, p)\right).\end{aligned}$$

Hence if $\phi(q, p)$ is decreasing in p, then the integrand is concave in p, it then follows from ranking theorem II that the optimal q^{**} for a mean preserving spread $\tilde{p}$ of $\overline{p}_n$ will decrease.

(D) If $C(q)$ is strictly convex then for any q^0 and $q^1 \neq q^0$

$$C(q^1) > C(q^0) + C'(q)(q^1 - q^0).$$

It follows that

$$0 = C(0) > C(q) + C'(q)(0 - q).$$

Define $z = pq - qC(q)$. Then

$$\begin{aligned}\phi(q, p) &= [pq - qC'(q)]\frac{v''(pq - C(q))}{v'(pq - C(q))} \\ &= [pq - C(q) + C(q) - qC'(q)]\frac{v''(pq - C(q))}{v'(pq - C(q))} \\ &= z\frac{v''(z)}{v'(z)} + [C(q) - qC'(q)]\frac{v''(z)}{v'(z)} \\ &= -R(z) + [C(q) - qC'(q)]A(z)\end{aligned}$$

The bracketed expression is positive. Therefore under the assumptions of IRRA and DARA the right-hand side is a decreasing function of z. As z is an increasing function of p it follows that $\phi(q, p)$ is a decreasing function of p. Hence, by (C), output per firm declines.