The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 2: Risk Bearing: The Optimum of the Individual

2.1 The Risk Bearing Optimum: Basic Analysis

Solution 2.1.1.

(A) For (i), $\lambda_1(1,0,0)+\lambda_2(0,1,0)+\lambda_3(0,0,1) = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3 = 0$, hence they are linearly independent. For (ii) as well, the only solution to $\lambda_1(1,1,1)+\lambda_2(1,4,0)+\lambda_3(0,7,1) = (0,0,0)$ is $\lambda_1 = \lambda_2 = \lambda_3 = 3$ hence it is linearly independent.

The combinations (iii) and (iv) are linearly dependent because

$$\begin{aligned} 2(0,2,3) - (0,4,6) &= (0,0,0) & \text{(iii)} \\ -2(1,3,2) - (4,0,5) + 3(2,2,3) &= (0,0,0) & \text{(iv)} \end{aligned}$$

(B) Let $Z = [z_{as}]$ be the matrix whose a-th row is the vector of asset a. Then we are solving (P_1, P_2, P_3) such that

$$Z \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} P_1^a \\ P_2^a \\ P_3^a \end{pmatrix} \tag{1}$$

Suppose Z is invertible then we can solve for P_i's for any given value of P_i^a's.

For (i), $Z = I$, the identity matrix, hence $P_1 = P_2 = P_3 = 1$. The three assets in (ii) are also linearly independent and we can solve equation (1) to obtain $P_1 = 0.6, P_2 =$

$0.1, P_3 = 0.3$. For (iii) and (iv), which are linearly dependent, there do not exist P_i's that solve equation (1).

Now we proceed to solve the corresponding market planes in the state-claim space. For (i), the endowment in state-claim terms is given by $(1, 1, 1)$, since $P_i = 1$ for all i, the wealth is 3. The market plane is then

$$c_1 + c_2 + c_3 = 3.$$

For (ii), the endowment in state-claim terms is given by $(2, 12, 2)$, and the wealth is thus given by $0.6(2) + 0.1(12) + 0.3(2) = 3$. Hence the market plane is then

$$0.6c_1 + 0.1c_2 + 0.3c_3 = 3.$$

For (iii), note that asset 3 is strictly better than asset 1. So, if indeed $P_1^A = P_2^A = P_3^A = 1$, then asset 1 will not be consumed. One need only consider assets 2 and 3. The market plane (is in the positive quadrant and) passes through $(3, 0, 3)$ and $(0, 12, 18)$.

For (iv), we must take account of the elements of the endowment position. Specifically, $(q_1, q_2, q_3) = (1, 1, 1)$ translates into $(c_1, c_2, c_3) = (7, 5, 10)$. The trading opportunity constraint passes through the endowment position.

(C) Graph omitted.

Solution 2.1.2.

(A) A portfolio of $q_1 = -1$, $q_2 = q_3 = 1$ leads to violations of non-negativity for cases (i) and (iv) below. The state incomes in each of the cases are: (i) $(-1, 1, 1)$, (ii) $(0, 10, 0)$, (iii) $(1, 2, 3)$, and (iv) $(5, -1, 6)$.

(B) For case (i), it is not feasible because the state-claim is $(-1, 3, 4)$. Case (ii) is feasible because the state-claim is $(0, 13, 4)$.

Solution 2.1.3.

(A) A feasible portfolio is given by $\sum_a q_a z_a$ such that $\sum P_a^A q_a = \sum P_a^A \overline{q}_a$. Hence the agent's maximization problem is given by

$$\max_{q_1,q_2,q_3} \frac{1}{3}\ln(q_1 z_{11} + q_2 z_{21} + q_3 z_{31}) + \frac{1}{3}\ln(q_1 z_{12} + q_2 z_{22} + q_3 z_{32}) + \frac{1}{3}\ln(q_1 z_{13} + q_2 z_{23} + q_3 z_{33})$$

s.t.

$$q_1 + q_2 + q_3 = 3.$$

For case (i), one can either solve the F.O.C.s or observe from symmetry that the optimal $(q_1, q_2, q_3) = (1, 1, 1)$, which induces the state-claim $(1, 1, 1)$.

For case (ii), the F.O.C.s are given by

$$\begin{aligned}
&\frac{1}{3(q_1+q_2)} + \frac{1}{3(q_1+4q_2+7q_3)} + \frac{1}{q_1+q_3} = \lambda \\
&\frac{1}{3(q_1+q_2)} + \frac{4}{3(q_1+4q_2+7q_3)} = \lambda \\
&\frac{7}{3(q_1+4q_2+7q_3)} + \frac{1}{3(q_1+q_3)} = \lambda \\
&q_1 + q_2 + q_3 = \lambda.
\end{aligned}$$

Solving the system of equations to obtain $(q_1, q_2, q_3) = (2, -1/3, 4/3)$, which induces the state-claim $(10/6, 10, 10/3)$.

(B) One can still solve the optimal (q_1, q_2, q_3) and compute the induced (c_1, c_2, c_3), but the state claim will not satisfy equation (2.1.6) in the text.

Solution 2.1.4.

(A), (B) The agent solves

$$\max_{\{c_s\}} \sum_{s\in S} \pi_s v(c_s)$$

s.t.

$$\sum_{s\in S} P_s c_s = \overline{W}.$$

The F.O.C with respect to state s is given by

$$\frac{\pi_s v'(c_s)}{P_s} = \lambda.$$

Hence for s, s' we have

$$\frac{\pi_s v'(c_s)/P_s}{\pi_{s'} v'(c_{s'})/P_{s'}} = 1. \tag{2}$$

When $v(c) = \ln c$, we then have

$$\frac{c_s}{c_{s'}} = \frac{\pi_s}{P_s}\frac{P_{s'}}{\pi_{s'}}. \tag{3}$$

(C) By (3), $c_s \geq c_{s'}$ for all s' if and only if

$$\frac{\pi_s}{P_s}\frac{P_{s'}}{\pi_{s'}} \geq 1 \tag{4}$$

for all s'. This defines the greatest c_s. The reverse inequality then defines the least c_s.

(D) Yes. Assume $v(c)$ is concave and twice differentiable. Then $v'(c)$ is decreasing. Hence $c_s \leq c_{s'}$ is equivalent to $v'(c_s)/v'(c_s') \leq 1$, which, by (2), is equivalent to (4).

Solution 2.1.5.

(A) The budget set is given by

$$\{q_1(100, 200) + q_2(200, 100)|150q_1 + 150q_2 = 150\}.$$

Substituting in the budget constraint $q_2 = 1 - q_1$ we get

$$(c_1, c_2) = (200 - 100q_1, 100 + 100q_1),$$

which is equivalent to $c_1 + c_2 = 300$.

(B) Since the market is complete, we can solve for the state claim price $(p_1, p_2) = (1/2, 1/2)$ from (A) and equivalently optimize over (c_1, c_2). The maximization problem is then

$$\max_{c_1, c_2} \pi(-e^{-c_1}) + (1-\pi)(-e^{-c_2})$$

s.t.

$$c_1 + c_2 = 300.$$

The F.O.C.s are given by

$$\begin{aligned} c_1 - c_2 &= \ln \frac{\pi}{1-\pi} \\ c_1 + c_2 &= 300 \end{aligned}$$

Hence

$$c_1^* = 150 + \frac{1}{2} \ln \frac{\pi}{1-\pi}.$$

(C) This follows from the expression in (A).

(D) When π becomes small, c_1 becomes negative infinity, and q_1^* becomes positive infinity. The agent thinks state 1 is unlikely to happen, hence he will invest in asset 1 for which state 2 yield is better than state 1 yield.

2.2 Choosing combinations of mean and standard deviation of income

Solution 2.2.1.

(A) Job 1 is the lottery $G_1 = (1, 3; 0.5, 0.5)$. Job 2 is the lottery $G_2 = (0, 2, 4; 1/9, 7/9, 1/9)$. We have $\mu_1 = 2, \sigma_1 = 1$, $\mu_2 = 2, \sigma_2 = 8/9$. Nevertheless,

$$E[v(G_1)] = 0.5\sqrt{1} + 0.5\sqrt{3} = 1.37 > E[v(G_2)] = \frac{7}{9}\sqrt{2} + \frac{1}{9}\sqrt{4} = 1.32$$

(B) The two lotteries are symmetric about the same mean and their skewness is zero. Hence, skewness cannot explain the preferences for G_1 and G_2. However, note that

$$v''''(c) = \frac{-25}{16} c^{-7/2} < 0$$

and

$$E[(G_1 - \mu_1)^4] = 1 < E[(G_2 - \mu_2)^4] = \frac{32}{9},$$

Hence if we use Taylor series to represent the utility function $v(c) = \sqrt{c}$ we can see that G_1 has a higher utility because its fourth moment around the mean is smaller.

Solution 2.2.2.

(A) The first identity follows from simple algebra. For the second one, we can apply the first identity to see that

$$\begin{aligned} Ev(c) &= -\int_{-\infty}^{\infty} e^{-Ac} \frac{1}{\sqrt{2\pi}} exp\left\{ -\frac{1}{2} \left(\frac{c-\mu}{\sigma} \right)^2 \right\} dc \\ &= -e^{-A(\mu - \frac{1}{2}A\sigma^2)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} exp\left\{ -\frac{1}{2\left(\frac{c-\mu+A\sigma^2}{\sigma} \right)^2} \right\} dc \\ &= -e^{-A(\mu - \frac{1}{2}A\sigma^2)}. \end{aligned}$$

(B) As $U(\mu, \sigma) = \mu - \frac{1}{2}A\sigma^2$ is a monotonic transformation of $Ev(c) = -e^{-A(\mu - \frac{1}{2}A\sigma^2)}$, it induces the same preferences on the set of lotteries.

Let the riskless asset be money, and the price of the risky asset be p, the endowed wealth be $\overline{W}$. A portfolio is then given by $W - pq_2 + q_2 z$, where z is a random variable with mean μ and standard deviation σ. Hence it's mean and standard deviation as a function of q_2 is given by

$$\begin{aligned} \mu(q_2) &= \overline{W} - pq_2 + q_2\mu \\ \sigma(q_2) &= q_2\sigma. \end{aligned}$$

Hence

$$U(\mu(q_2), \sigma(q_2)) = W - pq_2 + q_2\mu - \frac{1}{2}Aq_2^2\sigma^2.$$

The F.O.C. with respect to q_2 is then

$$-p + \mu - Aq_2\sigma^2 = 0.$$

Rearrange to get

$$q_2 = \frac{\mu - p}{A\sigma^2},$$

which is independent of $\overline{W}$. Assume $\mu > p$ (otherwise no one buys the risky asset), then q_2 is decreasing in A. When the expenditure on the risky asset exceeds $\overline{W}$ he will borrow money to buy the risky asset.

Solution 2.2.3.

(A) A portfolio is given by $\kappa\overline{W}z_a + (1-\kappa)\overline{W}z_b$. Hence

$$\mu(\kappa) = \kappa\overline{W}\mu_a + (1-\kappa)\overline{W}\mu_b$$

and

$$\sigma^2(\kappa) = \kappa^2\overline{W}^2\sigma_{aa} + 2\kappa(1-\kappa)\overline{W}^2\sigma_{ab} + (1-\kappa)^2\overline{W}^2\sigma_{bb}$$

(B) Note that

$$\begin{aligned} \mu(\kappa) - \overline{W}\mu_a &= (1-\kappa)\overline{W}(\mu_b - \mu_a) \\ \mu(\kappa) - \overline{W}\mu_b &= \kappa\overline{W}(\mu_a - \mu_b) \end{aligned}$$

Hence

$$\begin{aligned} \sigma^2(\kappa) &= \kappa^2\overline{W}^2\sigma_{aa} + 2\kappa(1-\kappa)\overline{W}^2\sigma_{ab} + (1-\kappa)^2\overline{W}^2\sigma_{bb} \\ &= \left(\kappa^2\overline{W}^2\sigma_{aa} + 2\kappa(1-\kappa)\overline{W}^2\sigma_{ab} + (1-\kappa)^2\overline{W}^2\sigma_{bb}\right)\frac{(\mu_a-\mu_b)^2}{(\mu_a-\mu_b)^2} \\ &= \left((\mu(\kappa)-\overline{W}\mu_b)^2\sigma_{aa} - 2(\mu(\kappa)-\overline{W}\mu_a)(\mu(\kappa)-\overline{W}\mu_b)\sigma_{ab} + (\mu(\kappa)-\overline{W}\mu_b)^2\sigma_{bb}\right)\frac{1}{(\mu_a-\mu_b)^2}. \end{aligned}$$

The claim thus follows.

(C) With $\sigma(\kappa) = A^{1/2}$, we have

$$\frac{d\sigma(\kappa)}{d\mu(\kappa)} = \frac{1}{2}A^{-1/2}\left(2(\mu(\kappa) - \overline{W}\mu_b)\sigma_{aa} - 2(2\mu(\kappa) - 2\overline{W}(\mu_a + \mu_b))\sigma_{ab} + 2(\mu(\kappa) - \overline{W}\mu_a)\sigma_{bb}\right)$$

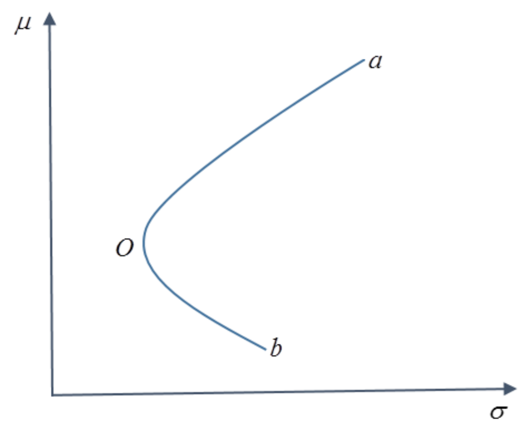

Figure 1: Ex 2.2.3(C)

If the agent puts all the wealth in asset b, then $\kappa = 0$, so $\mu(\kappa) = \overline{W}\mu_b$. Thus

$$\left.\frac{d\sigma(\kappa)}{d\mu(\kappa)}\right|_{\kappa=0} = \frac{1}{2}A^{-1/2}(4\overline{W}\mu_a\sigma_{ab} + 2\overline{W}(\mu_b - \mu_a)\sigma_{bb}) < 0$$

Which means that increasing shares of asset a can reduce σ and also increase μ (as $\mu_a > \mu_b$). The locus of feasible μ and σ is depicted in Figure 2.2.3(C). The point a corresponds to $\kappa = 1$ (i.e., 100% investment in asset a) and the point b corresponds to $\kappa = 0$.

(D) The optimal portfolio will lie on the efficient frontier, the region Xa in Figure 2.2.3(C).

Solution 2.2.4.

(A) If the agent hold only asset 1, he can buy 1.5 units, hence he can get $a_1 = (0, 1.5)$. If the agent hold only asset 2, he can buy $1.5/0.46 = 3.26$ units, hence he can get

$a_2 = (9.78, 3.26)$. If the agent hold only asset 3, he can buy $1.5/0.04 = 37.5$ units, hence he can get $a_3 = (150, 37.5)$.

The points a_1, a_2, and a_3 are depicted in Figure 2.2.4.

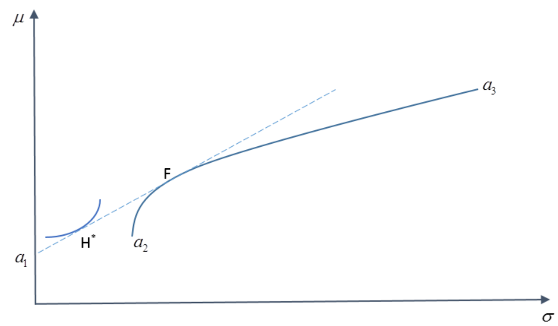

Figure 2: Ex 2.2.4

(B) Suppose $(q_1, q_2, q_3) = (1, 1, 1)$ is the optimal portfolio, H^*. This has a mean of 3=1+1+1 and variance of $25 = 3^2 + 4^2$. In the $\sigma\mu$-plane it translates to $H^* = (\sigma, \mu) = (5, 3)$. The individual's optimal portfolio is

$$\frac{1}{1.5}a_1 + \frac{0.46}{1.5}a_2 + \frac{0.04}{1.5}a_3$$

Hence, the mutual fund is

$$0.92a_2 + 0.08a_3,$$

which is $F = (15, 6)$ in the $\sigma\mu$-plane. The individual spends one-third of his income on the mutual fund.

(C) The price of risk reduction is the slope of the budget line, which passes through $(0, 1.5)$ and $(5, 3)$, hence is $3/10$. The MRS at H^* equals the slope by the tangency condition, hence it is also $3/10$.

2.3 State-Dependent Utility

Solution 2.3.1.

(A) For any c, it is easy to see that $v_W(c) > v_L(c)$ for all cases.

(B),(C) Since the odds are fair, in meeting the first-order conditions the individual in each case will set:

$$v'_W(\bar{c} + b) = v'_L(\bar{c} - b).$$

Straightforward computation then leads to:

- (i) $b = 60$: You bet on the home team and so evidently continue to prefer that the home team win.
- (ii) $b = 0$: You do not bet, so you still prefer that the home team win.
- (iii) $b = -33.1$ (approximately): You bet against the home team, but nevertheless still prefer that the home team win.
- (iv) $b = -25$: You bet against the home team, but now you are indifferent as to which team wins.

In (i), the marginal utility for winning is higher than that when losing, hence you bet for winning. In (ii), the marginal utility for winning or losing are the same, hence you do not bet. In (iii) and (iv), the marginal utility for losing is higher than winning, hence you bet against winning.

Solution 2.3.2.

(A) Let

$$8\sqrt{c - 20} = 5(c - 56)^{2/3},$$

we see that when $c = 120$ both the left hand side and the right hand side equal 80. As both utility functions are monotone, it means that when $c > 120$ the agent will choose to live in the second suburb and when $c < 120$ the agent will choose to live in the first suburb.

(B) If the outcome is $c = 181$ the agent will choose the second suburb, and if the outcome is $c = 56$ the agent will choose the first suburb. Hence

$$Ev(c) = 0.5 \times 5(181 - 56)^{2/3} + 0.5 \times 8\sqrt{56 - 20} > 80 = v(120)$$

(C) Note that around $c = 120$, the marginal utility of c is increasing, where the utility is given by

$$u(c) = \max\{8\sqrt{c - 20}, 5(c - 56)^{2/3}\}.$$

Hence the agent will like to gamble. The optimal fair gamble, $[c_1, c_2; p, 1 - p]$, where $pc_1 + (1 - p)c_2 = 120$, is given by the following graph.

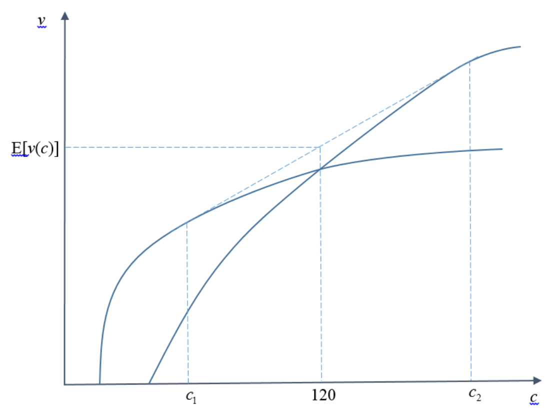

Figure 3: Ex 2.3.2(C)

(D) If a larger amount of money can help one jump to a situation with higher marginal utility of consumption, then one might want to gamble even if one's marginal utility for consumption in either state is locally decreasing.

Solution 2.3.3.

(A) The willingness to pay z for any p is defined as

$$z = \max_k \{k | pv(\bar{c} - k) \geq p_0 v(\bar{c})\}.$$

Since $v(c)$ is increasing, we have

$$pv(\bar{c} - z) = p_0 v(\bar{c}).$$

(B),(C) As

$$p = \frac{p_0 v(\bar{c})}{v(\bar{c} - z)}.$$

we have

$$\frac{dp}{dz} = \frac{p_0 v(\bar{c}) v'(\bar{c} - z)}{[v(\bar{c} - z)]^2}.$$

Furthermore,

$$\frac{d^2 p}{dz^2} = \frac{-p_0 v(\bar{c}) v''(\bar{c} - z)[v(\bar{c} - z)]^2 + 2v(\bar{c} - z) p_0 v(\bar{c})[v'(\bar{c} - z)]^2}{[v(\bar{c} - z)]^4} > 0$$

hence the shape is convex.

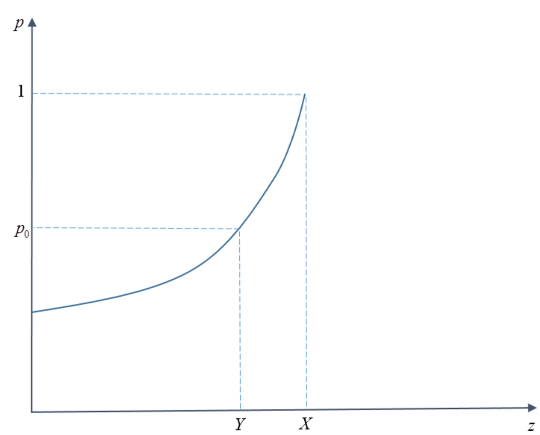

Figure 4: Ex 2.3.3(B)

(D) No. For example, consider $v(c) = \sqrt{c}$ we have

$$X = (1 - p_0^2)\overline{c}$$

and

$$Y = \left(1 - (\frac{p_0}{\frac{1+p_0}{2}})^2\right)\overline{c}.$$

If $p_0 = 0.5$ then

$$\frac{X}{Y} = \frac{27}{20} < 2.$$

(E) When p tends to p_0, the willingness to pay z will tend to 0, hence it follows from (B) that the value he places on his life is

$$\frac{p_0 v'(\overline{c})}{v(\overline{c})}.$$

This measures the increase in the probability of living that is worth a unit dollar; hence the reciprocal of this is the value of life, which is the same as what is in the text.