The Analytics of Information and Uncertainty Answers to Exercises and Excursions

Chapter 12: Information transmission, acquisition, and aggregation

12.1 Strategic information transmission and delegation

12.1.1 Strategic information transmission

Solution 12.1.1.1. To Pareto rank equilibrium, we first note that in ex-post, depending on the signal realized, players in the uninformative equilibrium may be better off than the partially informative equilibrium or vice versa. However, in ex-ante, we can unambiguously Pareto rank the equilibrium.

Since $-E[(x-b-s)^2|r_i] = -(E[(x-s)^2|r_i] - 2bE[x-s|r_i] + b^2)$ and that the receiver will choose $x_i = E[s|r_i]$ in equilibrium whenever r is received, the players' ex-ante expected utility is simply

$$EU^R = -\sum_{i=1}^{n} E[(x_i-s)^2|r_i]P(r_i)$$
$$EU^S = -\sum_{i=1}^{n} E[(x_i-s)^2|r_i]P(r_i) - b^2.$$

Hence both sender and receiver have the same ex-ante preference over different equilibria.

Since it is evident that

$$-\int_0^1 (s-0.5)^2 ds > -\int_0^{0.3} (s-0.15)^2 ds - \int_{0.3}^1 (s-0.65)^2 ds,$$

the partially informed equilibrium is the Pareto superior one.

Solution 12.1.1.2.

(A) Suppose s is uniformly distributed on (a_1, a_2). Then the receiver's utility is

$$-E[(x-s)^2],$$

which is minimized at $x = E[s] = (a_1 + a_2)/2$.

Alternatively, one can directly compute the expectation as a function of x to get

$$E[U(x,s)|x \in (a_1,a_2)] = \frac{1}{(a_2-a_1)}\left(x^2(a_2-a_1) - x(a_2^2-a_1^2) + \frac{a_2^3-a_1^3}{3}\right).$$

The FOC will then be $2x = a_2 + a_1$.

(B) The utility is then

$$-E[(s-E[s])^2] = -\frac{(a_2-a_1)^2}{12}.$$

Here we directly apply the formula of the variance of a uniform random variable on (a_1, a_2).

Solution 12.1.1.3.

(A) If $s = a$, the sender is indifferent if

$$-\left(\frac{a}{2} - a - b\right)^2 = -\left(\frac{1+a}{/}2 - a - b\right)^2.$$

Since it must be $a/2 < a + b < (1+a)/2$, the above equation implies

$$b + \frac{a}{2} = \frac{1}{2} - \frac{a}{2} - b.$$

Rearrange to get the desired expression.

(B) Consider the following strategy profile and belief:

- Strategy

  The receiver plays $a/2$ if he receives $r \leq r_1$, and plays $(a+1)/2$ if he receives $r > r_1$. The sender plays r_1 if $x \leq a$, plays r_2 if $x > a$.

- Belief

  The receiver believes $s \sim U(0, a)$ if he receives $r \leq r_1$. The receiver believes $s \sim U(a, 1)$ if he receives $r > r_1$.

We need to show the proposed strategy is sequentially optimal and the belief is sequentially consistent. For sequential optimality, if $s \in [0, 0.5 - 2b]$, the sender prefers $a/2$ to $(1+a)/2$ hence sending r_1 is optimal. Given the belief when r_1 is received, it is optimal for the receiver to choose $a/2$. A similar reasoning applies to the case $s \in (0.502b, 1]$. This shows sequential optimality.

To check sequential consistency of the receiver's belief, consider the following sender's completely mixed strategies indexed by ϵ. When $s \leq a$,

$$r_\epsilon(s) = \begin{cases} r_1 & \text{prob} \quad 1 - \epsilon - \epsilon^2 \\ U(0, r_1) & \text{prob} \quad \epsilon \\ U(r_1, 1) & \text{prob} \quad \epsilon^2 \end{cases}$$

When $s > a$,

$$r_\epsilon(s) = \begin{cases} r_2 & \text{prob} \quad 1 - \epsilon - \epsilon^2 \\ U(0, r_1) & \text{prob} \quad \epsilon^2 \\ U(r_1, 1) & \text{prob} \quad \epsilon \end{cases}$$

That is, when $s \leq a$, the mixed strategy plays r_1 with probability $1 - \epsilon - \epsilon^2$ and plays the uniform mixed strategy $U(0, r_1)$ on the interval $[0, r_1]$ with probability ϵ, and so forth.

It then suffices to show that the receiver's posterior belief induced by the sequence of strategy $r_\epsilon(s)$ converges to the belief we proposed as ϵ tends to zero. To this end, we first compute the conditional density of r given s. By the definition of $r_\epsilon(s)$, when $s \leq a$,

$$P(r_s(\epsilon) \leq r|s) = \begin{cases} \epsilon \frac{r}{r_1} & r < r_1 \\ 1 - \epsilon^2 & r = r_1 \\ 1 - \epsilon^2 + \epsilon^2 \frac{r - r_1}{1 - r_1} & r > r_1 \end{cases}$$

Hence[1]

$$f(r|s) = \begin{cases} \frac{\epsilon}{r_1} & r < r_1 \\ \frac{\epsilon^2}{1 - r_1} & r > r_1. \end{cases}$$

Similarly, when $s > a$ we have

$$f(r|s) = \begin{cases} \frac{\epsilon^2}{r_1} & r < r_1 \\ \frac{\epsilon}{1 - r_1} & r > r_1. \end{cases}$$

Now we are in a place to compute the conditional density of r given s by Bayes theorem: When $r < r_1$ and $s \leq a$,

$$\begin{aligned} f_s(s|r) &= \frac{f_r(r|s) f_s(s)}{\int_0^1 f_r(r|s) f_s(s) ds} \\ &= \frac{\epsilon / r_1}{\epsilon a / r_1 + (1 - a)\epsilon^2 / r_1} \end{aligned}$$

Letting $\epsilon \to 0$ we get $f_s(s|r) = 1/a$ when $s \leq a$. hence when $r < r_1$ is observed the receiver's belief converges to $U(0, a)$.

Similarly, when $r > r_1$ and $s > a$,

$$f_s(s|r) = \frac{\epsilon / (1 - r_1)}{a\epsilon^2 / (1 - r_1) + (1 - a)\epsilon / (1 - r_2)},$$

Letting $\epsilon \to 0$ we get $f_s(s|r) = 1/(1 - a)$ when $s > a$. hence when $r > r_1$ is observed the receiver's belief converges to $U(a, 1)$.

For the limit of the belief induced by $r_\epsilon(s)$ on the equilibrium path, we can not compute through the conditional density $f_r(r|s)$ because the it does not exist. Instead, we compute the conditional

[1] $P(r_\epsilon(s) \leq r | s \leq a)$ is not differentiable at $r = r_1$, but it is differentiable everywhere else.

distribution. For $x \le a$

$$\begin{aligned} P(s \le x | r_1) &= \frac{P(r_1 | s \le x) P(s \le x)}{P(r_1 | s \le x) P(s \le x) + P(r | s > x) P(s > x)} \\ &= \frac{(1 - \epsilon - \epsilon^2) x}{(1 - \epsilon - \epsilon^2) x + (1 - \epsilon - \epsilon^2)(a - x)} \\ &= \frac{x}{a}. \end{aligned}$$

Hence the receiver's belief in the equilibrium path is correct for every ϵ, which of course converges to $U(0, a)$. A similar argument applies to $P(s \le x | r_2)$. This proves sequential consistency.

Solution 12.1.1.4.

(A) Suppose there exists a such that when $s = a$ the sender is indifferent between $a/2$ and $(1 + a)/2$. Then by Exercise 3(A) $a + 2b = 0.5$. But $b > 0.25$ then implies $a < 0$, which is impossible. Hence such a does not exist.

(B) Suppose there are 3 or more intervals, $[0, a_1), [a_1, a_2), \ldots$. Then the receiver's optimal action when he observes r_1 is $a_1/2$, and when he observes r_2 is $(a_1 + a_2)/2$, and so on. When $s = a_1$, the sender must be indifferent between $a_1/2$ and $(a_1 + a_2)/2$. So

$$b + \frac{a_1}{2} = \frac{a_2}{2} - \frac{a_1}{2} - b,$$

or

$$2b + a_1 = \frac{a_2}{2}.$$

If $b > 0.25$, then $a_2/2 > 0.5 + a_1$, or $a_2 > 1$, which is impossible. Hence there is no equilibrium with three or more intervals.

Solution 12.1.1.5.

(A) If the receiver believes $x \sim U(a_i, a_{i+1})$, the receiver is going to choose $x = (a_i + a_{i+1})/2$. The belief induced by the sender's strategy is exactly that when r_i is observed, the receiver believes $x \sim U(a_i, a_{i+1})$. Hence for each i, the receiver takes $(a_i + a_{i+1})/2$ if r_i is received.

(B) When $s = a_i$, the sender must be indifferent between sending r_i or r_{i+1}, given that the receiver follows the strategy described in (A). Hence

$$a_i + b - \frac{a_{i-1}}{2} - \frac{a_i}{2} = \frac{a_{i+1} + a_i}{2} - a_i - b.$$

Rearrange to get

$$a_{i+1} = 2a_i - a_{i-1} + 4b.$$

(C) By plugging in $a(i) = a_1 i + 2i(i-1)b$ to the difference equation, we can see that it is satisfied and hence it is a class of solution.

(D) Since a_1 can be made arbitrarily small, $N(b)$ will be the largest integer i such that

$$2i(i-1)b < 1.$$

Thus, for any $b > 0$, $N(b)$ must be finite. To solve $N(b)$, note that $N(b) = N$ satisfies

$$2bN^2 - 2bN - 1 < 0$$
$$2bN^2 + 2bN - 1 > 0.$$

By using the quadratic formula we can get two intervals, and $N(b)$ lies within their intersection. One can show that the length of the intersected interval is 1, hence it must contain an integer. Thus $N(b)$ is just the floor function applied to the right end point of the interval, or

$$N(b) = \left\lfloor -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} \right\rfloor.$$

Hence $N(b)$ is decreasing in b.

12.2 Strategic information acquisition

12.2.1 Efficient information acquisition

Solution 12.2.1.1.

(A) Let H_1 denote the event $\{(H_1, L_2), (H_1, H_2)\}$, H_2 denote the event $\{(L_1, H_2), (H_1, H_2)\}$. Then

$$\begin{aligned}
P(H_1|m = l) &= \frac{P(m = l|H_1)P(H_1)}{P(m = l|H_1)P(H_1) + P(m = l|L_1)P(L_1)} \\
&= \frac{(0.5 - p)0.5}{(0.5 - p)0.5 + (0.5 + p)0.5} \\
&= 0.5 - p. \\
P(H_2|m = l) &= \frac{P(m = h|H_2)P(H_2)}{P(m = h|H_2)P(H_2) + P(m = h|L_2)P(L_2)} \\
&= \frac{(0.5 + p)0.5}{(0.5 + p)0.5 + (0.5 + p)0.5} \\
&= 0.5.
\end{aligned}$$

(B)

$$\begin{aligned}P(m=h) &= P(m=h|H_1)P(H_1)+P(m=l|L_1)P(L_1)\\ &= (0.5-p)0.5+(0.5+p)0.5\\ &= 0.5.\end{aligned}$$

Solution 12.2.1.2.

(A) Suppose player 2 bids truthfully, that is, bids $5+10p_2$ when he observes $m_2=h$ and bids $5-10p_2$ when he observes $m_2=l$. Suppose buyer 1 purchases p_1 and gets $m_1=h$. Then it is a best response of player 1 to bid $5+10p_1$: If $p_1<p_2$, he wins only if $m_2=l$, but in this case, since player 2 purchases more info than player 1, player 1's expected value conditional on that he wins is also $5-10p_2$, so whatever player 1's bid is, his expected payoff is zero. If $p_1\geq p_2$, then player 1 always wins, and his payment is only $5+10p_2$ while his expected utility is $5+10p_1$. If he bids lower than $5+10p_2$ then he makes a loss, any bid higher than $5+10p_2$ generates the same expected payoff. A similar argument can be made for $m_1=l$.

(B) Suppose $p_2=0$. Then the optimal p_1 is 0.25 by the result in the text. Suppose $p_1=0.25$. Then buyer 1 subs either 2.5 or 7.5 with probability 0.5 respectively. Buyer 2's expected surplus as a function of p_2 is given by

$$\Pi_2(p_2)=\begin{cases}0.5(0.5(5+10p_2-2.5)+0.5(5+10p_2-7.5))=5p_2 & p_2>0.25\\ 0.5(0.5(7.5-2.5))=1.25 & p_2=0.25\\ 0.5^2(5+10p_2-2.5)+0.5^2(5-10p_2-2.5) & \end{cases}$$

Since

$$\frac{d}{dp}(5p-10p^2)=5-20p<0$$

when $p>0.25$, $p_2=0$ is the best response to $p_1=0.25$.

(C) Given (p_1,p_2) where $p_1>p_2$,

$$\begin{aligned}S(p_1,p_2) &= S(p_1,p_2|(l,l))P(l,l)+S(p_1,p_2|(l,h))P(l,h)+S(p_1,p_2|(h,l))P(h,l)+S(p_1,p_2|(h,h))P(h,h)\\ & \frac{1}{4}\left((5-10p_2)+(5+10p_2)+(5+10p_1)+(5+10p_1)\right)\\ &= 5+5p_1.\end{aligned}$$

The social cost is $10p_1^2+10p_2^2$. Hence social surplus is maximized at $(p_1,p_2)=(0.25,0)$.

If $p_1 = p_2 = p$, then $S(p_1, p_2) = \frac{1}{4}(5 - 10p + 5 + 10p + 5 + 10p + 5 + 10p) = 5 + 5p$. But the social cost is $10p^2 + 10p^2$, hence the surplus is less than the situation where $p_1 > p_2$. Similarly, if $p_1 < p_2$ then the social surplus is maximized at $(p_1, p_2) = (0, 0.25)$.

12.2.2 Overinvestment in information

Solution 12.2.2.1.

(A) Since

$$\begin{aligned} P(l|H_1) &= P(l|\{(H_1, L_2), (H_1, H_2)\}) \\ &= \frac{P(l \cap (H_1, L_2)) + P(l \cap (H_1, H_2))}{P((H_1, L_2), P(H_1, H_2))} \\ &= 0.5(0.5 - kp) + 0.5(0.5 - p) \\ &= 0.5 - 0.5(k+1)p. \end{aligned}$$

and similarly $P(l|L_1) = 0.5 + 0.5(1+k)p$, Bayes' theorem implies

$$P(H_1|l) = \frac{P(l|H_1)P(H_1)}{P(l|H_1)P(H_1) + P(l|L_1)P(L_1)} = 0.5 - 0.5(k+1)p.$$

(B)

$$P(l) = P(l|H_1)P(H_1) + P(l|L_1)P(L_1) = 0.5.$$

Solution 12.2.2.2.

(A)

Message	B1 Bid	B2 Bid	B1 Surplus	B2 Surplus	Social Surplus
l	5-5(1+k)p	5-5(1-k)p	0	10kp	5-5(1-k)p
h	5+5(1+k)p	5-5(1-k)p	10p	0	5+5(1+k)p.

(B) By the table in part(A)

$$\Pi_1(p) = \frac{1}{2}(5 + 5(1+k)p - (5 - 5(1-k)p)) = 5p$$

so it is optimal when

$$5 = 20p$$

or $p = 0.25$.

(C) Social surplus is given by

$$S(p) = 0.5(5 - 5(1-k)p) + 0.5(5 + 5(1+k)p) = 5 + 5kp,$$

which is maximized when $5k = 20p$, or $p = 0.25k < 0.25$. Hence bidder 1 overinvests in information. Social surplus does not change.

Solution 12.2.2.3. Since

$$\frac{d\Pi_1(p)}{dp} = 2.5(1+k), \frac{dS}{dp} = 5k,$$

the private optimum p^* and social optimum p^s satisfy

$$c'(p^s) = 5k$$
$$c'(p^*) = 2.5(1+k).$$

Since $2.5(1+k) > 5k$ and $c''(p) > 0$, we have $p^* > p^s$.

Solution 12.2.2.4.

(A) Suppose no one gathers information, then the only mutual best response is $(5, 5)$. But given that $b_2 = 5$, bidder 1 can purchase some info, say p_1, and bid $5 + 0.1$ if $m = h$, bid 0 if $m = 2$, his payoff will be

$$\frac{1}{2}(5 + 5(1+k)p - (5 + 0.1)) = \frac{1}{2}(5(1+k)p - 0.01) > 0$$

for p large enough.

Hence in equilibrium bidder 1 purchases some information.

Suppose $p > 0$ and that $(b_1^*(l), b_1^*(h), b_2^*)$ is a Nash equilibrium. Then when $m = h$, bidder 2's expected value of the object is $b_2^* \leq 5 + 5(1-k)b$, so $b_2^* \leq 5 + 5(1-k)p$. And in this situation the best response of bidder 1 is to bid slightly higher than b_2^*, since bidder 1's expected value of the object is $5 + 5(1+k)p > 5 + 5(1-k)p \geq b_2^*$.

Suppose $m = l$, then bidder 1's expected value of the object is $5 - 5(1+k)p < 5 - 5(1-k)p$. So in equilibrium $b_1^*(l) \leq 5 - 5(1+k)b$. In equilibrium it can not be that $b_2^* \leq b_1^*(l)$ because under such strategy profile his payoff is always zero, while if he deviates to some $b_1^*(l) < b_2 < 5 - 5(1-k)p$ he has some chance to earn positive payoff. Also, it can not be $b_1^*(l) < b_2^* < 5 - 5(1+k)p$ otherwise bidder 1 will deviate to some $b_2^* < b_1(l) < 5 - 5(1+k)p$. So in equilibrium $b_2^* > 5 - 5(1+k)p$. But then when bidder 1 observes l, bidder 1 will not want to win. Hence if $(b_1^*(l), b_1^*(h), b_2^*)$ is an equilibrium, it must be that $b_1^*(h) = b_2^* + 0.01$ and $b_1^*(l) < b_2^*$.

(B) To characterize NE for different p, suppose b_2^* is an NE strategy(and thus $b_1^*(l) = b_2^* - 0.01$, $b_1^*(h) = b_2^* + 0.01$). Then for bidder 2, he is worse off by deviating to $b_2^* + 0.2, b_2^* + 0.1, b_2^* - 0.1, b_2^* - 0.2$. It suffices to consider only these four deviations. Also, if he deviates to $b_2^* + 0.02$ then he wins with probability 1, so he does not get new information upon winning. Thus it is necessary that

$$\begin{aligned}
0.25(5 + 5(1-k)p - b_2^* - 0.01) + 0.5(5 - 5(1-k)p - b_2^* - 0.01) &\leq 0.5(5 - 5(1-k)p - b_2^*) \\
(5 - b_2^* - 0.02) &\leq 0.5(5 - 5(1-k)p - b_2^*) \\
0.25(5 - 5(1-k)p - b_2^* + 0.01) &\leq 0.5(5 - 5(1-k)p - b_2^*) \\
0 &\leq 0.5(5 - 5(1-k)p - b_2^*).
\end{aligned}$$

These inequalities boil down to

$$5 + 5(1-k)p - 0.03 \leq b_2^* \leq 5 - 5(1-k)p - 0.01.$$

Hence, for NE to exist, it must be $10(1-k)p \leq 0.02$. Note that in this case,

(C) Suppose $b_2^* = 5 - 5(1-k)p - 0.01$, Buyer 1's expected payoff as a function of p where $p \leq 0.02/10(1-k)$ is then

$$0.5(5 + 5(1+k)p - 5 - 5(1-k)p) = 5kp,$$

so he will still buy $p = 5k/20$, the same inefficient amount as the second price auction outcome.

12.3 Information cascades

Solution 12.3.1.

(A) Alex's expected payoff for the information is

$$frac12E[V|h_1] = p,$$

hence he buys the message if and only if $p - c \geq 0$.

(B) Given that Alex purchases information, Bev knows $m = h_1$ if Alex adopts, and $m = l_1$ if Alex does not adopt. Suppose Alex adopts, then the expected utility for Bev to Adopt is $E[V|h_1] = 2p$. If she purchases the information, since $E[V|h_1 l_2] = 0$, her expected payoff is then

$$P(h_2|h_1)E[V|h_1 h_2].$$

Since

$$P(h_2|h_1) = \frac{P(h_1h_2|V=1)P(V=1) + P(h_1h_2|V=2)P(V=2)}{P(h_1|V=1)P(V=1) + P(h_1|V=2)P(V=2)} = (0.5+p)^2 + (0.5-p)^2$$

and that

$$\begin{aligned} E[V|h_1h_2] &= (P(V=1|h_1h_2) - P(V=-1|h_1h_2)) \\ &= \frac{(0.5+p)^2 - (0.5-p)^2}{(0.5+p)^2 + (0.5-p)^2}, \end{aligned}$$

the expected utility for purchasing information given Alex adopts is still $2p$. If Alex does not adopt, then $E[V|l_1l_2] < 0$, $E[V|l_1h_2] = 0$, so Bev will not purchase if it's costly, since she will never earn a positive payoff. Hence whenever the information has a positive cost, Bev will not purchase it.

For agents after Bev, if $c > 0$, since Bev won't purchase, they are in the same situation as Bev, so they will not purchase.

(C) More likely. In fact, if $c > 0$, then Bev will follow Alex, and Cede will follow Bev, and so on ad infinitum.

Solution 12.3.2. Conditional on $V = 1$, the probability that the fifth individual accepts is

$$P(A|V=1) = P(A|V=1,4H)P(4H|V=1) + P(A|V=1,3H)P(3H|V=1) + P(A|V=1,2H)P(2H|V=1) + P(A|V=1,$$

where nH denotes that the fifth individual observes n H signals. Since the individual will not adopt if he observes $1H$ and $0H$, and adopts with probability 0.5 when he observes $2H$, and adopts with probability one otherwise, using the assumption that the signals are independent conditional on V, we obtain

$$\begin{aligned} P(A|V=1) &= P(4H|V=1) + P(3H|V=1) + 0.5P(2H|V=1) \\ &= (0.5+p)^4 + 4(0.5+p)^3(0.5-p) + 0.5(6)(0.5+p)^2(0.5-p)^2 \end{aligned}$$

One can check that for $p = 0.3$, $P(A|V=1) = 0.104$.

More generally, given $V = 1$, if there are $2n$ signals, the $2n$+1-th person is correct with probability

$$P(A|V=1) = \sum_{i=n+1}^{n} P(iH|V=1) + 0.5P(nH|V=1),$$

and each $P(iH|V=1)$ is just the probability that $2n$ independent binomial random variables success i times.

12.4 The Condorcet Jury Theorem

Solution 12.4.1.

(A) Since

$$P(s=s_a|X_i=0)=\frac{P(X_i=0|s=s_a)P(s=s_a)}{P(X_i=0|s=s_a)P(s=s_a)+P(X_i=0|s=s_b)P(s=s_b)}=\frac{9}{13},$$

we have

$$E[u(A,s)|X_i=0]=\frac{9}{13}u>E[u(B,s)|X_i=0]=\frac{4}{13}u. \tag{1}$$

Similarly,

$$P(s=s_a|X_i=1)=\frac{1}{7},$$

so

$$E[u(B,s)|X_i=1]=\frac{6}{7}u>E[u(A,s)|X_i=1]=\frac{1}{7}u. \tag{2}$$

The sincere voting strategy is then vote A if $X_i=0$, vote B if $X_i=1$.

(B) Yes, by (1) and (2).

(C) Suppose voter 2,3 vote informatively. Then vote 1 will evaluate his payoff as if he is pivotal. That is, as if $X_2+X_3=1$. When $X_1=0$, voter 1 considers the situation where $\sum_{i=1}^3 X_i=1$, and he compute

$$\begin{aligned}
P(s=s_a|\sum_{i=1}^3 X_i=1) &= \frac{P(\sum_{i=1}^3 X_i=1|s=s_a)P(s=s_a)}{P(\sum_{i=1}^3 X_i=1|s=s_a)P(s=s_a)+P(\sum_{i=1}^3 X_i=1|s=s_b)P(s=s_b)}\\
&= \frac{q_a^2(1-q_a)}{q_a^2(1-q_a)+q_b(1-q_b)^2}\\
&= \frac{81}{177}
\end{aligned}$$

So $E[u(A,s)|\sum_{i=1}^3 X_i=1]=(81/177)u<E[u(B,s)|\sum_{i=1}^3 X_i=1]=(96/177)u$. That is, he thinks B is better even if $X_1=0$. Since $P(s=s_b|\sum_{i=1}^3 X_i=2)>P(s=s_b|\sum_{i=1}^3 X_i=1)$, it follows that he thinks B is even better if $X_1=1$. So voter 1's best response is to always choose B, which is not informative.

Solution 12.4.2.

(A) By Exercise 1(C), the best response to two informative voters is to always vote B. Hence the remaining step to show NE is to show the best response to one informative and one always vote for B voter is to be informative. Suppose bidder 1 always votes for B, bidder 2 is informative. Then bidder 3 is pivotal when bidder 2 vote for A, or $X_2 = 0$. He can not get any information about X_1 from voter 1's behavior. Voter 3 computes

$$P(s = s_a | X_2 + X_3 = 0) = \frac{q_a^2}{q_a^2 + (1 - q_b)^2} = \frac{81}{97}$$
$$P(s = s_a | X_2 + X_3 = 1) = \frac{(1 - q_a) q_a}{(1 - q_a) q_a + (1 - q_b) q_b} = \frac{9}{33}.$$

So when $X_3 = 0$ and $X_2 = 0$, voter 3 should vote for A, and if $X_3 = 1, X_2 = 0$, voter 3 should vote for B. In sum, voter 3's best response is to be informative.

(B)

$$\begin{aligned} P(A|s = s_a) &= P(A|s = s_a, X_i = 0) P(X_i = 0 | s = s_a) + P(A|s = s_a, X_i = 1) P(X_i = 1 | s = s_a) \\ &= r_0 q_a + r_1 (1 - q_a) \end{aligned}$$

$$\begin{aligned} P(B|s = s_b) &= P(B|s = s_b, X_i = 0) P(X_i = 0 | s = s_b) + P(B|s = s_b, X_i = 1) P(X_i = 1 | s = s_b) \\ &= (1 - r_0)(1 - q_b) + (1 - r_1) q_b \end{aligned}$$

(C) Let $(r_0, r_1) = (0.815, 0)$ be a symmetric strategy profile. As before, the voter should consider the situations when they are pivotal. For player 1, suppose $X_1 = 0$. Consider the event $(X_1 = 0, A, B)$, meaning that player 1's signal is 0, player 2 votes for A and player 3 votes for B. Since A depends on X_2 and B depends on X_3, $X_1 = 0, A$ and B are conditionally independent given s. So

$$\begin{aligned} P(s = s_a | (X_1 = 0, A, B)) &= \frac{P(X_1 = 0, A, B | s = s_a) P(s = s_a)}{P(X_1 = 0, A, B | s = s_a) P(s = s_a) + P(X_1 = 0, A, B | s = s_b) P(s = s_b)} \\ &= \frac{P(X_1 = 0 | s = s_a) P(A | s = s_a) P(B | s = s_a)}{P(X_1 = 0 | s = s_a) P(A | s = s_a) P(B | s = s_a) + P(X_1 = 0 | s = s_b) P(A | s = s_b) P(B | s = s_b)} \\ &= 0.5 \end{aligned}$$

Hence when player 1 observes $X_1 = 0$, he is indifferent between voting A and B. When $X_1 = 1$,

$$\begin{aligned} P(s = s_a | (X_1 = 1, A, B)) &= \frac{P(X_1 = 1 | s = s_a) P(A | s = s_a) P(B | s = s_a)}{P(X = 1 | s = s_a) P(A | s = s_a) P(B | s = s_a) + P(X_1 = 1 | s = s_b) P(A | s = s_b) P(B | s = s_b)} \\ &= 0.12, \end{aligned}$$

hence when $X_1 = 1$ and voter 1 is pivotal, he prefers B, so $r_1 = 0$(Never vote A) is a best response.