# The Analytics of Information and Uncertainty
# Answers to Exercises and Excursions

## Chapter 11: Long-run relationships and the credibility of threats and promises

### 11.1 The multi-period Prisoners' Dilemma

**Solution 11.1.1.** Yes. What player 1 suggests is a better equilibrium or both sides of the subgame they are currently playing. If player 2 buys the argument, they can then switch to a better equilibrium than the grim outcome of defect forever.

**Solution 11.1.2.**

(A) The equilibrium path of the Grim NE gives a utility of

$$U_i = \frac{f}{1-\delta}$$

to a player.

If a player deviates, his opponent will play Defect forever, so the best the player can get after his deviation is $g$ forever. Hence the utility for deviation is

$$U_i^d e + \frac{\delta g}{1-\delta}.$$

There is no incentive to deviate from the equilibrium path if and only if

$$e + \frac{\delta g}{1-\delta} < \frac{f}{1-\delta},$$

which rearranges to

$$\delta > \frac{e-f}{e-g}.$$

Since $e - f < e - g$, we do not need additional restrictions on the parameters to guarantee that both playing Grim is a NE for sufficiently large $\delta$.

(B) Consider the following strategy: Play $C$ in the first round. If the opponent plays $D$ in the previous round, play $D$ for $t$ rounds. Then switch back to $C$. For sufficiently large $\delta$, there exists $t$ large enough such that this is also a NE strategy. The deviation payoff is

$$e + \delta\left(\frac{1-\delta^t}{1-\delta}\right)g + \frac{\delta^{t+1}}{1-\delta}f.$$

Compare this with the payoff when both players follows the strategy:

$$f + \delta\left(\frac{1-\delta^t}{1-\delta}\right) f + \frac{\delta^{t+1}}{1-\delta} f.$$

If $\delta$ is large enough, $(f-g)\delta/(1-\delta) > (e-f)$. For $t$ large enough, $(1-\delta^t)$ is close to one. Hence the payoff one can get from staying with the strategy is higher than that from deviation.

**Solution 11.1.3.**

(A) One can easily see that the two pure strategy NEs are $(x_2^1, x_2^2)$ and $(x_3^1, x_3^2)$.

(B) Consider the strategy suggested in the hint: Play $(x_1^1, x_1^2)$ until the last three rounds. Play $(x_2^1, x_2^2)$ in the last three rounds if there is no previous deviation. Once a player deviates, play $(x_3^1, x_3^2)$ thereafter.

Suppose there are $t$ periods remaining. If $t \geq 4$, the payoff for staying with the strategy is greater than deviation if and only if

$$(t-3)5 + (3)4 \geq 10 + (t-1)2$$

if and only if

$$t \geq \frac{11}{3}.$$

Hence if $t \geq 4$, the proposed strategy constitutes an NE that supports cooperation until the last three rounds.

## 11.2 Subgame-perfect equilibria in infinitely repeated games

**Solution 11.2.1.**

(A) To show that the proposed strategy profile is a NE, it suffices to show that no one want to deviate from the path induced by the strategy. That is, no one wants to deviate from COOP. The problem is that one shot deviation principle does not hold for Nash equilibrium. That is, the claim that a strategy profile is a Nash equilibrium if there is no profitable one shot deviation from the path induced by the profile, is false. Hence in theory we need to consider all possible deviations. However, we can eliminate many of them. First, suppose a player deviates in COOP. Then it is clear that in PUNISH he should follow the original strategy rather than deviate again, since deviation means a negative and more zeros ahead but follow means he can get at least 2 in 5 rounds. Hence, the only thing the player should consider is whether he should deviate when the state is COOP. Therefore, to compare the payoff of deviation or staying with the proposed strategy, we only need to compare the payoffs one can get in a cycle of 6 periods. If one deviates, one gets 6 in the first period and 0 in the next five. If one stays with the current strategy, one gets

$$U_i = \frac{1-\delta^6}{1-\delta}2 = \frac{1-0.8^6}{1-0.8}2 = 7.3 > 6.$$

Hence there is no incentive to deviate from the COOP state.

(B) We have explained why deviations in PUNISH are not profitable. Since the subgames of the repeated game can be categorized to either COOP or PUNISH under the proposed strategy profile, we have shown that the strategy is a Nash equilibrium in every subgame. Hence it is a SPNE.

(C) If $\delta$ is greater, it only means that $U_i$ is greater, hence less incentive to deviate. To get the smallest $\delta$ to support the strategy profile as SPNE, one solves

$$\frac{1-\delta^6}{1-\delta}2 < 6.$$

Using any numerical tools available, we can get $\delta > 0.709$. [1]

(D) If $\tau = 4$, then we solve

$$\frac{1-\delta^5}{1-\delta}2 < 6,$$

which solves to$\delta > 0.74$. Intuitively, if the length of punishment is shorter than the players need to be more patient for the strategy profile to be NE.

(E) No, if $\tau = 2$, then deviation gets 6 while staying with the current strategy gets

$$2 + \delta 2 + \delta^2 2,$$

which is strictly less than 6 for any $\delta < 1$. If $\tau = 1$ then deviation gets 6 while staying with the current strategy gets $2 + \delta 2$. Hence when $\tau \leq 2$ this strategy does not constitute a NE for any $\delta$.

## 11.3 The folk theorem for infinitely repeated games

**Solution 11.3.1.** First, when the number of players are large, it may be more difficult for a player to monitor all other players' previous behavior since he needs to keep track of a large amount of information. Second, sometimes punishment needs coordination of players, which will be difficult to achieve if there are too many players.

[1] A convenient equation solver is provided by http://www.wolframalpha.com/

**Solution 11.3.2.**

(A) When $a = 14, c = 2$, the total output that maximizes total profit is $q = (a - c)/2 = 6$. Suppose each firm plays $q = 3$, then the per period profit is $(14 - 6)3 - (2)3 = 18$ Consider the standard minmax strategy: Firm $i$ plays $q_i^t = 6$ in time $t$ if $(q_i^{t'}, q_{-i}^{t'}) = (3, 3)$ for $t' < t$, otherwise firm $i$ plays the minmax strategy $q_i = 12$ forever.

Fix $\delta \in (0, 1)$. The above strategy is a NE if and only if

$$\frac{18}{1-\delta} > \max_q (9 - q)q = \frac{81}{4},$$

which solves to

$$\delta > 1/9.$$

(B) Suppose $\delta = 1/9$. Then one can not support any other $q_1 + q_2 = 6$ than the symmetric one as a long run Nash equilibrium. This is because if $q_i < 3$, then the profit will be

$$(9 - q)q < 18,$$

thus $\frac{(9-q)q}{1-1/9} < 81/4$, which means deviation is better than cooperation.

(C) The lower $\delta$ is, the smaller the set of achievable outcomes is. This is because cooperation is the sacrifice of immediate profits (deviation) for long-run profits. If $\delta$ is small, it means people are more impatient, and value the present more than the future, which makes it more difficult to support cooperation.

**Solution 11.3.3.**

(A) The minmax output with asymmetric cost for player $i \in \{1, 2\}$ is

$$q_i^* = a - c_{-i}.$$

which implies $\min_{q_i} \max_{q_{-i}} U_{-i}(q_i, q_{-i}) = 0$. Let $(U_1, U_2) > (0, 0)$ be single stage payoffs for firm 1 and 2 supported by the output vector $(q_1, q_2)$. Let $U_i'$ be the maximum gain from deviation by firm $i$ when firm $-i$ plays $q_i$ in a single period. Then for sufficiently large $\delta$,

$$\frac{U_i}{1-\delta} > U_i',$$

hence minmax punishment can still support cooperation as a Nash equilibrium.[2]

[2] To support $(U_1, U_2)$ as a SPNE outcome, one can still construct strategies as in Section 11.3.

(B) Let $(U_1, ..., U_n)$ be single period payoffs supported by the output vector $(q_1, ..., q_m)$. Consider the following strategy for firm $i$:

- A. Play $q_i$ in period $t$ if for previous periods $(q_1, ..., q_n)$ is played, otherwise enter phase B.
- B. Play $q_i^* = a - c$ for $\tau$ periods.
- C. Play $q_i$ again after $\tau$ periods if in these periods $(q_1^*, ...q_n^*)$ is played, otherwise begin phase B again.

Let $U_i^o$ be the single period negative payoff for firm $i$ when $(q_1^*, ..., q_n^*)$ is played. Then using exactly the same argument as in Section 11.3, we can show that for sufficiently large $\delta$ there exists $\tau$ such that there is no one shot profitable deviation under the proposed strategy profile.

(C) To ensure that no firm gets profit when the costs are asymmetric, we can just let the firms play $q_i^* = a$. This output drives price to zero so no firm can make any profit. This may however require a larger $\delta$ to support SPNE since the payoffs for playing $(a, ..., a)$ become more negative than before.

## 11.4 The role of chivalry

**Solution 11.4.1.**

(A) First, we define the n-person prison dilemma game. Let $I = \{1, ..., n\}$ be the set of players. Each player's strategy set is $\{C, D\}$. The payoff for player $i$ choosing $C$ is $f(C|k)$, where $k \in \{0, ..., n-1\}$ is the number of players choosing $C$. Similar for $f(D|k)$. Assume that the following holds:

- $f(C|k) < f(D|k)$ for $k \in \{0, ..., n-1\}$.
- $f(C|n-1) > f(D|0)$.
- $f(C|k)$ and $f(D|k)$ are increasing in $k$.

For a normalization, assume $f(D|0) = 0$. This generalizes the two person prisoners' dilemma in the obvious way.

Now let us add perturbations to player types. Suppose each player has a probability $\epsilon$(independently) of being chivalrous, as defined in the text. Similar to the text, we calculate player $i$'s payoff for mimicking the chivalrous players. Suppose there are $\tau$ periods left. With probability $\epsilon^{n-1}$ all other players are chivalrous. The present value of always mimicking the chivalrous players would then be

$$f(C|n-1)(1 + \delta + ...\delta^{\tau-1}).$$

With probability $(1-\epsilon^{n-1})$ some of them are not chivalrous. The worst possible situation is that all other players play $D$ forever, then player $i$ has a payoff of $f(C|0)$ in the current period and $f(D|0) = 0$ thereafter. It follows that player 1's expected payoff from mimicking the chivalrous strategy is bounded from below by

$$\epsilon^{n-1} f(C|n-1)(1+\delta+...+\delta^{\tau-1}) + (1-\epsilon^{n-1}) f(C|0)$$

Since the payoff to deviation is at most $f(D|n-1)$, mimicking is better as long as

$$\epsilon^{n-1} f(C|n-1)(1+\delta+...+\delta^{\tau-1}) + (1-\epsilon^{n-1}) f(C|0) > f(D|n-1).$$

which holds for sufficiently large $\tau$ if

$$\frac{1}{1-\delta} > \frac{f(D|n-1) - f(C|0)}{f(C|n-1)\epsilon^{n-1}} + \frac{f(C|0)}{f(C|n-1)}. \tag{1}$$

(B) Rewrite the function $f(\cdot|k)$ as $f_n(\cdot|k)$ to emphasize its dependency on the total number of players. Assume $f_n(\cdot|n)$ is constant over $n$. Then by (1) one sees that the larger $n$ is, the larger the right hand side is, since $\epsilon < 1$. This implies that as the number of players increase, it will require a larger $\delta$ to support cooperation. The intuition is that large number means a deviation to $D$ is more likely to appear, hence playing $C$ becomes more risky.

**Solution 11.4.2.** In the Cournot game with market demand $p = a - q$ and cost $c$, the symmetric Nash equilibrium is $q^N = (a-c)/3$, where each firm gets a payoff of $(a-c)^2/9$. Let $(q_1, q_2) = (q, q)$ be the output that gives a profit of $\Pi(q,q) > (a-c)^2/9$ for each firm. Define chivalrous behavior as playing $q$ until the other firm deviates, then play the Nash equilibrium $q^N = (a-c)/3$. Suppose there are $\tau$ periods left. For firm 1, mimicking gives it

$$\epsilon\Pi(1+\delta+...+\delta^{\tau-1}) + (1-\epsilon)(\frac{5(a-c)^2}{12} + \frac{(a-c)^2}{9}(\delta+...+\delta^{\tau-1})).$$

The expected payoff from deviation is

$$\max_{q_i} \epsilon(a-c-q_i-q)q_i + (1-\epsilon)(a-c-q_i-\frac{a-c}{3})q_i + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9} = A + \frac{\delta}{1-\delta}\frac{(a-c)^2}{9}$$

Hence, as long as

$$\frac{\epsilon\Pi}{1-\delta} + (1-\epsilon)\frac{5}{12}(a-c)^2 + (1-\epsilon)\frac{(a-c)^2}{9}\frac{\delta}{1-\delta} > A + \epsilon\frac{(a-c)^2}{9}\frac{\delta}{1-\delta}$$

for sufficiently large $\delta$, then there exists $\tau$ such that mimicking is better than deviating.

Rearrange the inequality to get

$$\epsilon\frac{1}{1-\delta}(\Pi - \delta\frac{(a-c)^2}{9}) + (1-\epsilon)\frac{5}{12}(a-c)^2 > A$$

Since $\Pi > (a-c)^2/9$ by assumption, the inequality indeed holds for sufficiently large $\delta$. This completes the proof.

## 11.5 Building a reputation

**Solution 11.5.1.**

(A) Since Mild fight is dominant for type $M$ and Hard fight is dominant for type $H$, once Mild fight(Hard fight) is observed the posterior probability for the firm being type $H(M)$ will be zero. Hence the Bayesian updating will be the same as in the text after the firm chooses his action in the first period.

Let $\pi_t$, $t \in \{M, H\}$ be the threshold posterior probability that the firm is of type $t \in \{M, H\}$ such that the entrant is indifferent between enter and out. That is,

$$\pi_M \frac{-1}{2} + (1-\pi_M)2 = 0$$
$$\pi_H(-1) + (1-\pi_H)2 = 0$$

Or $\pi_M = 4/5$, $\pi_H = 2/3$.

Assuming $\epsilon_H = \epsilon_M = \epsilon$. Equation (11.5.4) then shows that

$$\frac{\ln \epsilon}{\ln \pi_M} - 1 \leq m^* \leq \frac{\ln \epsilon}{\ln \pi_M}$$
$$\frac{\ln \epsilon}{\ln \pi_H} - 1 \leq m^{**} \leq \frac{\ln \epsilon}{\ln \pi_H}.$$

Since $2 \ln 4/5 \approx \ln 2/3$, it follows that $m^* \approx 2m^{**}$.

(B) If the chain store fights mildly, then

$$\begin{aligned} PV^* &= (1 + \delta + ... + \delta^{m^*-1})\frac{1}{2} + 3\frac{\delta^{m^*}}{1-\delta} \\ &= \frac{1-\delta^{m^*}}{1-\delta}\frac{1}{2} + 3\frac{\delta^{m^*}}{1-\delta}. \end{aligned}$$

If the chain store fights hard, then

$$\begin{aligned} PV^{**} &= (1 + \delta + ... + \delta^{m^{**}-1})0 + \frac{\delta^{m^{**}}}{1-\delta}3 \\ &= 3\frac{\delta^{m^{**}}}{1-\delta}. \end{aligned}$$

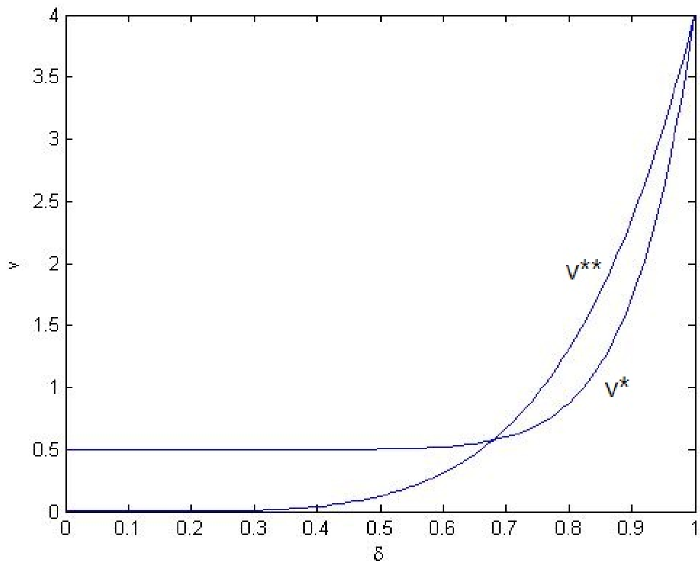


Figure 1: Plot for $v^*, v^{**}$

Since $v^*(1) = v^{**}(1) = 3 > 1$ and

$$\left.\frac{dv^*}{d\delta}\right|_{\delta=1} = \frac{7}{2}m^* > 4m^{**} = \left.\frac{dv^{**}}{d\delta}\right|_{\delta=1},$$

when $\delta < 1$ and is close to 1, $v^{**} > v^* > 1$. Which means

$$PV^{**} > PV^* > \frac{1}{1-\delta}.$$

This shows that mimicking type $H$ is optimal.

(C) Recall that

$$v^* = (1 - \delta m^*)\frac{1}{2} + \delta^{m^*}3$$
$$v^{**} \approx 3\delta^{m^{**}}.$$

Let $v^* = v^{**}$ and use $m^* = 2m^{**}$ we have

$$\frac{1}{2} + \frac{7}{2}(\delta^{m^{**}})^2 = 4\delta^{m^{**}},$$

hence $\delta^{m^**} = 1$ or $1/5$. But when $\delta^{m^**} = 1/5$, $v^{**} = 4/5 < 1$. Since $v^*{*}$ and $v^*$ are both increasing in $\delta$, for $\delta^{m^{**}} > 1/5$, $v^{**} > v^*$, and when $\delta^{m^{**}} < 1/5$, $v^* < 1$. Hence Mild fight is never optimal.

(D) When $\theta = 1/2$, for $\delta^{m^{**}} < 1/5$ but sufficiently colse to $1/5$, we have $1/2 < v^{**} < v^*$ by the analysis in (C). Hence this holds for $\theta$ around some neighborhood of $1/2$.

(E) If $\epsilon_M > \epsilon_H$, $m^*$ will be smaller. This will improve the payoff of Mild fight, hence may make $v^* > v^{**}$ when $\delta$ is large enough.

**Solution 11.5.2.**

(A) It is a dominant strategy for a type 2 entrant to enter. Hence if the prior probability of type 2 entrant is high, the weak chain store will just choose match, because mimicking the strong type will not change type 2 entrant's behavior anyway.

(B) Let $p$ be the prior probability of type 2 entrant. Suppose the firm chooses to mimic the strong firm. Then entry will happen for $\overline{m}$ periods as defined in equation (11.5.4). However, once this period passes, there is still a probability of $p$ in each period that the entrant is of type 2 and will choose enter. Hence the payoff for mimicking is then

$$\frac{\delta^{\overline{m}}(1-\delta^{n-\overline{m}})}{1-\delta}3(1-p).$$

Mimicking is better if and only if

$$\frac{\delta^{\overline{m}}(1-\delta^{n-\overline{m}})}{1-\delta}3(1-p) > \frac{(p/2+(1-p))(1-\delta^n)}{1-\delta}.$$

Observe that even if $p$ is zero, there are still some parameter values that makes the inequality untrue. Hence whether the chain store fights still depends on other parameter values, not just $p$ alone.

(C) No. In the last period the firm who already convinced the type 1 entrants that it is strong will choose Match anyway since there is no future for the entrants to update accordingly.

**Solution 11.5.3.**

(A) Assume $n = \infty$. First, if the firm chooses Match, then the entrant will conclude that the firm is not strong. Hence the entrant's posterior becomes $(0, \beta, 1-\beta)$. Let $\overline{m}$ be the maximum number of entry for the entrant when in the previous history the firm plays Undercut. Then the payoff for mimicking the strong firm is

$$U_S = \frac{1-\delta^{\overline{m}-1}}{1-\delta} + \frac{\delta^{\overline{m}}}{1-\delta}3.$$

If both types of firm chooses Match, then there is no way for the future entrants to tell whether the firm is less weak or weak by simply observing actions(not payoffs). The entrants will then always choose enter. The firm's payoff is then

$$U_{LW} = \frac{2}{1-\delta}$$
$$U_W = \frac{1}{1-\delta}.$$

Hence when $U_S > U_{LW}$, both types of firms will choose to mimic.

(B) By (A) the only possible situation is that the less weak type chooses Match but the Weak type mimics the Strong. Suppose this is the case then by Bayesian updating the type who chooses Match will be identified with probability one by the entrants. The entrants' belief when he observes match will then be $(0, 0, 1)$, that is, he puts probability one that the firm is the less weak type. Hence the entrants are indifferent between Enter and Out. If the entrants choose Out, then the Weak type will want to deviate to Match, so this is not possible in equilibrium. If the entrants choose Enter, then the weak type will not deviate. Hence such kind of equilibrium is possible.[3]

**Solution 11.5.4.** If the entrants choose out from the beginning, then the weak firm will deviate to Match, hence this is not an equilibrium. Since choosing $(Match, In)$ whenever the firm is weak is also an equilibrium when $\epsilon$ is small, by the analysis in the text we need $\overline{m}$ consecutive Fights to convince the entrants that this is not possible.

**Solution 11.5.5.**

(A) If the entrant chooses out, he will not get any new information. Hence $H^{t-1}$ and $(h_t = O, H^{t-1})$ provides the same information.

(B)

$$\begin{aligned} P(S|x_{1t} = U, x_{2t} = I, H^{t-1}) &= \frac{P(S, x_{1t} = U|x_{2t} = I, H^{t-1})}{P(x_{1t} = U|x_{2t} = I, H^{t-1}} \\ &= \frac{P(S|H^{t-1})}{P(x_{1t} = U|H^{t-1})}. \end{aligned}$$

Hence, for all history $H^t$ that contains purely $(U, I)$,

$$P(S|H^t) = \frac{P(S|H^{t-1})}{P(x_{1t} = U|H^{t-1})}. \tag{2}$$

(C) It follows from (2) and that $P(x_{1t} = U|H^{t-1}) \leq 1$.

(D) Note that $\overline{\pi}$ is the same as before, irrelevant of whether the actions are ex-post observable. Then a successive application of (2) to each $t$ implies

$$P(S|H^t) \geq \left(\frac{1}{\overline{\pi}}\right)^m \epsilon.$$

Since $\overline{m}$ is the largest integer that

$$\left(\frac{1}{\overline{\pi}}\right)^m \epsilon \leq 1,$$

it is the same as before.

[3]The maximum number of entry $\overline{m}$ is endogenous, namely, under different equilibria $\overline{m}$ could be different.

(E) If the weak chain store chooses fight with probability 1 every period, the best response for the entrants is Out. But then the weak chain store will deviate to Not fight. Hence pure strategy can not be equilibrium.