# The Analytics of Information and Uncertainty
# Answers to Exercises and Excursions

## Chapter 10: Market Institutions

### 10.1 Posted-price Markets

**Solution 10.1.1.** Inserting everything to equation (10.1.1) in the text to get

(A) $p = \frac{4}{9}$

(B) $p = \frac{1}{2}$

(C) $p = \frac{1}{\sqrt{3}}$.

The prices are increasing from (A) to (C) because

$$1 - F_A(p) < 1 - F_B(p) < 1 - F_C(p)$$

for all $p$. Hence, for all $p$, the probability that the valuation are above $p$ in (C) is higher than that of (B), which in term is higher than that of (A). If the valuation is likely to be high, the seller should post a higher price to take advantage of it.

**Solution 10.1.2.**

(A) First we compute the buyer's cut-off valuation $v$ such that he is indifferent between accepting and rejecting the first period offer $p = 1/2$. It satisfies

$$v - \frac{1}{2} = \frac{1}{2}\left(v - \frac{1}{4}\right).$$

Hence $v = 3/4$. Then note that in the second round, since the offer is final, the buyer will accept if and only if $v \geq 1/4$. Hence buyer's best response is:

$$\begin{aligned}
&\text{Accept in the first round if} \quad v > \frac{3}{4}\\
&\text{Reject in the first round but accept in the second if} \quad v \in [1/4, 3/4]\\
&\text{Always reject if} \quad v < \frac{1}{4}.
\end{aligned}$$

(B) Given the buyer's best response, the seller will make

$$0.25 \times \frac{1}{2} + 0.5 \times \frac{1}{4} + 0 = \frac{1}{4}$$

If the seller simply uses the take-it-or-leave-it offer with $p = 1/2$, the buyer will accept if and only if $v > 1/2$, hence the seller can earn an expected profit of $1/4$. These two strategies are the same.

## 10.2 Auctions

### 10.2.1 Bidders' valuations are known

**Solution 10.2.1.1.**

(A) Given Bev bids according to the proposed way, denoted by $b_2(\lambda)$ Alex' expected utility for bidding $\beta^{r-1}$ is

$$\begin{aligned} U_1(\beta^{r-1}, b_2(\lambda)) &= P(b_2 < \beta^{r-1})(V_1 - \beta^{r-1}) + P(b_2 = \beta^{r-1})\frac{1}{2}(V_1 - \beta^{r-1}) \\ &= (p(r-1)\lambda + p/2)(V_1 - \beta^{r-1}). \end{aligned}$$

(B) Similar computation shows that if Alex bids $b_1 = \beta^k < \beta^{r-1}$ he gets

$$U_1(b_1, b_2(\lambda)) = (kp\lambda + p\lambda/2)(V_1 - \beta^k).$$

(C) Now observe that if $\lambda$ is sufficiently, small,

$$U_1(b_1, b_2(\lambda)) < \frac{p}{2}(V_1 - \beta^{r-1}) < U_1(\beta^{r-1}, b_2(\lambda))$$

for $b_1 < \beta^{r-1}$.

Intuitively, when $\lambda$ becomes small, it is more unlikely for Bev to bid below $\beta^{r-1}$, so playing $b_1 < \beta^{r-1}$ becomes unlikely to win.

Hence, given sufficiently small $\lambda$, we can let $p \to 0$ to see that $b_2(\lambda) \to \beta^r$.

(D) Consider the following bidding strategy $b_1(\lambda, p)$for Alex: Play $\beta^j, j = 0, ..., r-2, r+1, ...T$ with probability $\lambda p$, play $\beta^r$ with probability $p$, and $\beta^{r-1}$ with $1 - (T-1)\lambda p - p$. Then when $\lambda$ is small, if Bev bids $b_2 < \beta^r$ she will get a smaller expected utility than bidding $b_2 = \beta^r$. Then, since $p \to 0$, to have $\lim_p b_1(\lambda, p) = \beta^{r-1}$, (C) and (D) shows the bidding $(b_1, b_2) = (\beta^{r-1}, \beta^r)$ is trembling hand perfect.

**Solution 10.2.1.2.**

(A) Correct. In the first price auction, a higher deviation leads to a loss and a lower one still gets oneself zero utility. Hence bidding $V$ is a best response to everyone else all bid $V$. In an open ascending-price auction, if one quit earlier given others quit at $V+1$, one gets nothing. But staying a little while longer to beat others' $V$ gives you a loss. Hence staying until $V$ is again a best response for other bidders doing so.

(B) In the open ascending-price auction, it is better to bid $V$(stay in the auction when the price does not exceed $V$), because if the guy who plays $V$ trembles and thus quits early, the bidder who plays $V$ will gain. Hence the proposed equilibrium is not trembling hand perfect.

(C) Yes, one class of Nash equilibria is as follows: $k$ bidders bid up to $V$ while the rest $n-k$ bidders bid 0, for $1 \leq k \leq n$. However, the seller is always able to earn $V$ in any NE. This is because, if $(b_1, ..., b_n)$ is a bidding profile such that $\max\{b_i\} < V$, then any player can deviate to $\max b_i + \epsilon$ and make a profit. If $b_j = \max\{b_i\} > V$, then bidder $j$ will do better by deviating to any $b_j \leq V$. Hence in a Nash equilibrium we must have $\max\{b_i\} = V$. According to the rule of first price auction, the seller will then earn $V$ in any Nash equilibrium.

**Solution 10.2.1.3.**

(A) Without loss of generality, suppose $V_1 \leq V_2$. One class of Nash equilibria is as follows: Bidder 1 quits at any $v < V_1$, bidder 2 quits at any $v > V_1$. Then obviously no bidder wants to deviate, hence these are NEs.

(B) Let $F(x)$ be a distribution with support $[0, \infty)$. Let $b_i(\lambda)$ be the bidding rule such that with probability $1-\lambda$, $V_i$ is played and with probability $\lambda$ bidder $i$ bids according to $F$. Then $b_i(\lambda)$ is a completely mixed strategy which converges to $V_i$[1] as $\lambda$ tends to zero.

Given bidder $i$ bids according to $b_i(\lambda)$, bidding $b_j$ gives bidder $j$

$$U_j(V_j, b_i(\lambda)) = (1-\lambda)1_{V_j > V_i}(V_j - V_i) + \lambda E[V_j - V_i | V_i < b_j].$$

By observing the second term, one can see that is optimal for player $j$ to bid $b_j = V_j$.

To put it in words, since bidding truthfully in the open ascending price auction is a weakly dominant strategy, when the opponent uses a completely mixed strategy it will become a strict best response.

[1] Formally, to a degenerate random variable that puts probability one on $V_i$.

(C) No. If bidder $i$ bids $b_i < V_i$ and bidder $j$ trembles to $b_i < b_j < V_i$ then $i$ is forgoing a potential gain. If bidder $i$ bids $b_i > V_i$ and bidder $j$ trembles to $b_i > b_j > V_i$ then bidder $i$ is making a loss.

### 10.2.2 Bidders' valuations are independent and privately known

**Solution 10.2.2.1.** Let $V_i$ denote bidder $i$'s value, which is a uniform distribution on $[a, b]$. Suppose bidder 2 bids according to $b(v_2)$. Then bidder 1's expected utility for bidding $b_1 \leq (a + b)/2$ is[2]

$$U(b_1, b(V_2); v_1) = (v_1 - b_1)F(V_2 < 2b_1 - a) = (v_1 - b_1)\frac{2b_1 - 2a}{b - a},$$

which is concave. The FOC is then

$$2v_1 + 2a = 4b_1$$

Hence the optimal $b_1$ is

$$b_1(v_1) = \frac{a + v_1}{2}.$$

By symmetry, $b(V) = (a+V)/2$ are mutually best responses, hence constitutes a Nash equilibrium.

**Solution 10.2.2.2.** Let $\beta(v)$ be a symmetric equilibrium bidding strategy. The expected payoff for player $i$ with value $v$ when he bids $b$ and his opponent bids according to $\beta(v)$ is

$$(\beta^{-1}(b))^{\alpha}(v - b).$$

The first order condition with respect to $b$ is

$$\alpha(\beta^{-1}(b))^{\alpha-1}\frac{1}{\beta'(\beta^{-1}(b))}(v - b) - (\beta^{-1}(b))^{\alpha} = 0.$$

Since $\beta(v)$ is a NE, the FOC will be satisfied when $b = \beta(v)$. Hence,

$$\beta'(v)v^{\alpha} = \alpha v^{\alpha-1}(v - \beta(v)).$$

Rearrange to

$$\beta'(v)v^{\alpha} + \alpha\beta(v)v^{\alpha-1} = \alpha v^{\alpha}.$$

Since the left hand side equals to $(\beta(v)v^{\alpha})'$, integrate both sides to get

$$\beta(v)v^{\alpha} = \frac{\alpha}{\alpha + 1}v^{\alpha+1}.$$

Using $\beta(0) = 0$, we have

$$\beta(v) = \frac{\alpha}{\alpha + 1}v,$$

[2] Since bidder 2 never bids above $(a + b)/2$, it is without loss to restrict bidder 1's bidding to be below $(a + b)/2$.

which is linear in $v$.

The bid shading factor is

$$\frac{v - b(v)}{v} = \frac{1}{\alpha + 1},$$

which decrease with $\alpha$.

A higher $\alpha$ means the value is more likely to be high($v^{\alpha_1} > v^{\alpha_2}$ if $\alpha_1 > \alpha_2$.), hence one needs to bid more aggressively to win.

**Solution 10.2.2.3.**

(A) Suppose bidder $i$'s valuation $v$ is less than $r$. If he bids zero or any bid less than $r$ he always gets 0 If he bids higher than $r$, then given the other bidders bid less than his bid, he will win but make a loss, given the other bidders bid higher than his bid, he still gets 0. Hence $b = 0$ is a weakly dominant strategy for a bidder with value $v < r$.

Suppose now the bidder $i$ has $v > r$ and bids $b$. If $b > v$ then he will make a loss when $b > b_{-i} > v$, where $b_{-i} = \max_{j \neq i}\{b_j\}$ is the highest bid submitted by bidders other than $i$. For other values of $b_{-i}$ he gets the same payoff as bidding $b = v$. If $r < b < v$ then bidding $v$ is better when $b < b_{-i} < v$, in which case he pays $\max\{b_{-i}, r\} < v$. For other values of $b_{-i}$ he gets the same payoff as bidding $b = v$. If $b < r < v$, then he will always get zero regardless of his opponents' bids. Hence bidding $b = v$ is weakly dominant.

(B) Let $\beta^r(V)$ be the symmetric Nash equilibrium. Let the bidder's valuation be $v$. Suppose $v < r$. Then it is obvious that the optimal strategy is $b = 0$ since if he bids otherwise he either loses or needs to pays at least $r$ to win. Suppose $v > r$. Then his expected utility for bidding $b > r$ is

$$(v - b)P(\beta^r(V) < b).$$

The first order condition is then the same as exercise 2, which solves to

$$(v\beta^r(v))' = v.$$

However, because the differential equation is defined only for $v \in [r, 1]$, this time we integrate both sides from $r$ to $v$:

$$v\beta^r(v) - r\beta^r(r) = \frac{v^2 - r^2}{2}. \tag{1}$$

Finally, since we assumed $\beta^r(v)$ is differentiable in $[r, 1]$, it must be that $\beta^r(r) = r$. Substituting this to (1) to get

$$\beta^r(v) = 0.5v\left(1 + \frac{r^2}{v^2}\right)$$

(C) The expected payment of a bidder with value $v \geq r$ in a first price auction when both players play the symmetric NE is

$$m(v) = Pr(V < v)\beta^r(v) = 0.5(v^2 + r^2).$$

The expected payment of a bidder with value $v$ in a second price auction when his opponent bids truthfully is

$$\begin{aligned} m(v) &= Pr(V < v)\,\mathrm{E}[\max V, r | V < v] \\ &= r^2 + \int_r^v y dy \\ &= 0.5(v^2 + r^2). \end{aligned}$$

Since in both auctions the bidders have the same expected payment, the expected revenues of the seller are the same in both auctions.

**Solution 10.2.2.4.** First we consider the first price auction. Suppose $\beta(c)$ is an symmetric Nash equilibrium. Suppose the opponent bids according to $\beta(c)$. The expected payoff of a contractor to bid $p$ when his cost is $c$ is then

$$(p - c)P(\beta(C) > p) = (p - c)(1 - G(\beta^{-1}(p))).$$

Since we assumed $\beta(c)$ to be a Nash equilibrium, the first order condition

$$1 - G(\beta^{-1}(p)) + (c - p)g(\beta^{-1}(p))\frac{1}{\beta'(\beta^{-1}(p))} = 0$$

is satisfied when $p = \beta(c)$. Hence

$$1 - G(c) + \frac{(c - \beta(c))g(c)}{\beta'(c)} = 0.$$

Rearrange to

$$(\beta(c)(1 - G(c)))' = cg(c).$$

Hence

$$\beta(c) = \frac{1}{1 - G(c)} \int_0^c xg(x)dx.$$

Then consider the second price auction. We claim that bidding $c$ is a weakly dominant action for a contractor 1 with cost $c$. Let $p_2$ be the opponent's bid and $p_1 > c$. If $p_2 > p_1 > c$, then he wins and can get $p_2 - c$ no matter he bids $p_1$ or $c$. If $p_1 \geq p_2 > c$, then he is better off by bidding $c$. If $p_1 > c > p_2$ he loses anyway. Hence any bid $p_1 > c$ is dominated by bidding $c$. A similar argument shows that any bid $p_1 < c$ is dominated by $c$. Hence $c$ is weakly dominant, and truthful bidding is a dominant strategy equilibrium.

**Solution 10.2.2.5.**

(A) Since bidding truthfully is a weakly dominant strategy, it does not matter what probability distribution the opponent is using over his pure strategies. Hence risk preferences are irrelavent.

(B) The way to play an open descending auction is to decide a bid $b$ and buy the item when prices drops to $b$. Hence when all bids $b_1, ..., b_n$ are submitted, the highest bidder gets the object and pays his bid. Hence risk preferences are again irrelevant.

(C) If the price drops below his valuation, the lottery that he is faced with is losing the auction(get 0) or winning it but pay at a lower price. Hence he gets $pU(0) + (1-p)U(V-b)$. Hence a more risk averse person will want to avoid losing more, hence will decide to buy it when the current bid has not dropped too low.

(D) The first price sealed bid auction is equivalent to open descending auction, hence it follows from (C) that the buyers will bid more aggressively in a first price auction when they are risk averse. The open ascending price is equivalent to a second price auction, and since truth telling is dominant, their bidding behavior in these two auction forms will remain the same. Since first price auction and second price auction generate equivalent revenue when the agents are risk neutral, it must be that with risk averse agents, first price auction generate more revenue than the second price auction, which has the same revenue as the open ascending auction.

### 10.2.3 Optimal mechanisms

**Solution 10.2.3.1.**

(A) The assumption that the buyer's valuation is $V_1$ and it is better to report $V_1$ instead of $V_1'$ means

$$q(V_1)V_1 - x(V_1) \geq q(V_1')V_1 - X(V_1').$$

Hence

$$(q(V_1) - q(V_1'))V_1 \geq x(V_1) - X(V_1')$$

Since $V_1 > V_1'$ and $q$ is decreasing, the left hand side is negative. Hence

$$(q(V_1) - q(V_1'))V_1' > x(V_1) - x(V_1').$$

Rearrange to get

$$q(V_1)V_1' - x(V_1) > q(V_1')V_1' - x(V_1'),$$

which says that when the buyer's value is $V_1'$ he prefer to report $V_1$ than $V_1'$.

(B) This follows by interchanging $V_1$ and $V_1'$ in part (A).

**Solution 10.2.3.2.**

(A) Since the buyer with the highest valuation gets the object and the transfers between the buyer and the seller cancels out, the social gain is $E[\max V_1, V_2] = 2/3$. Seller's expected revenue is

$$E[\max\{\frac{V_1}{2}, \frac{V_2}{2}\}] = \frac{1}{3}.$$

For a buyer with valuation $V = v$ and bids $v/2$, his expected payoff is

$$F(v)(v - \frac{v}{2}) = \frac{v^2}{2},$$

so the ex-ante expected payoff of a buyer is

$$\int_0^1 \frac{v^2}{2} dv = \frac{1}{6}.$$

Finally note that the sum of buyers' expected payoff and the seller's expected revenue equals to the expected social gain.

(B) With a reserve price of 0.5 and a equilibrium bidding strategy with $b(0.25) = 0.25$, the expected social gain will be

$$P(\max\{V_1, V_2\} > 0.5) E[\max\{V_1, V_2\} | \max\{V_1, V_2\} > 0.5] = \int_{0.5}^1 2v^2 dv = \frac{7}{12}.$$

By Exercise 3 in this section, the expected payment of a bidder with valuation $v \geq r$ is

$$m(v) = 0.5(0.5^2 + v^2)$$

So the ex-ante expected payment of a bidder is

$$\int_{0.5}^1 \frac{1}{8} + \frac{v^2}{2} dv = \frac{10}{48}.$$

Therefore, firm's expected revenue is $2(10/48) = 5/12$.

The expected payoff of a buyer with value $v < 0.5$ is 0 and with $v \geq 0.5$ is

$$F(v)(v - b(v)) = v^2 - F(v)b(v) = v^2 - m(v),$$

hence the ex-ante expected payoff of a buyer is

$$\int_0 .5^1 v^2 - F(v)b(v) dv = \frac{1}{3} - \frac{1}{24} - \frac{5}{24} = \frac{1}{12}.$$

Finally note that the sum of ex-ante expected payoff of buyers and the firm's expected revenue equals the expected social gain.

### 10.2.4 Bidders' valuations are correlated

**Solution 10.2.4.1.**

(A) Assuming $b^f(\cdot)$ is increasing and $-i$ with type $t_{-i}$ bids $b(t_{-i})$. The expected payoff for a bidder with type $t_i$ to pretend his type is $x$ and bids $b^f(x)$ instead of $b^f(t_i)$ is then

$$\begin{aligned}\Pi(x,t_i) &= P(t_{-i}<x)(E[\frac{1}{2}(t_i+t_{-i})|t_{-i}<x]-b^f(x)) \\ &= \int_0^x (\frac{1}{2}t_i+\frac{1}{2}y)f(y)dy - F(x)b^f(x). \end{aligned} \tag{2}$$

(B) Taking first order condition of (2) with respect to $x$ to obtain

$$\frac{(}{1}2t_i+\frac{1}{2}x)f(x)-\frac{d}{dx}[F(x)b^f(x)]=0.$$

By the assumption that $b^f(x)$ is This equation is satisfied when $t_i=x$. Hence

$$xf(x)=\frac{d}{dx}[F(x)b^f(x)].$$

(C) Integrate both sides to get

$$\int_0^t xf(x)dx = F(t)b^f(t).$$

Therefore,

$$b^f(t)=\frac{1}{F(t)}\int_0^t xf(x)dx.$$

(D) First, we note that in the case where valuation is interdependent, bidding truthfully is not a dominant strategy in a second price auction. Suppose your type is 1 but the other player's type is 0. Then the object gives you a utility of 1/2. However, if your opponent bids 0.99, then you should not truthfully bid 1 since you will need to pay 0.99. However, we can show that bidding truthfully is a Nash equilibrium. That is, when your opponent bids truthfully your best response is also bidding truthfully.

To this end, suppose that player $-i$ bids truthfully, then the expected payoff for player $i$ with type $t_i$ to bid $x$ is

$$\begin{aligned}\Pi(x,t_i) &= P(t_{-i}<x)(E[\frac{1}{2}(t_i+t_{-i})-t_{-i}|t_{-i}<x] \\ &= \int_0^x \frac{1}{2}(t_i-y)f(y)dy.\end{aligned}$$

The first order condition is

$$(t_i-x)f(x)=0,$$

which is satisfied when $x=t_i$.

(E) The ex-ante expected payment for a buyer in (C) is

$$E[F(t)b^f(t)] = \int_0^1 \int_0^t xf(x)dxf(t)dt.$$

The ex-ante expected payment for a buyer in (D) is

$$E_{t_i}[P(t_i \geq t_{-i})E_{t_{-i}}[t_{-i}|t_i \geq t_{-i}]] = \int_0^1 \int_0^t xf(x)dxf(t)dt.$$

Hence the two auction forms yield the same revenue.

**Solution 10.2.4.2.**

(A) Assuming $b^f(\cdot)$ is increasing and $-i$ with type $t_{-i}$ bids $b(t_{-i})$. Since each buyer's type $t_i$ is i.i.d. uniform, the expected payoff for a bidder with type $t_i$ to pretend his type is $x$ and bids $b^f(x)$ instead of $b^f(t_i)$ is given by

$$\begin{aligned}\Pi(x, t_i) &= P(\max_{j\neq i}\{t_j\} < x)(E[\sum_{i=1}^{n} \frac{1}{n} t_i - b^f(x)| \max_{j\neq i}\{t_j\} < x]) \\ &= \frac{1}{n} t_i F^{n-1}(x) + \frac{n-1}{n} F^{n-2}(x) \int_0^{t_i} yf(y)dy - F^{n-1}(x)b^f(x) \\ &= \frac{1}{n} t_i x^{n-1} + \frac{n-1}{2n} x^n - x^{n-1} b^f(x). \end{aligned} \tag{3}$$

(B) The first order condition of (3) with respect to $x$ is

$$\frac{n-1}{n} t_i x^{n-2} + \frac{n-1}{2} x^{n-1} - \frac{d}{dx}[x^{n-1}b^f(x)] = 0,$$

which is assumed to be satisfied at $x = t_i(b^f(x)$ is a NE), so

$$\frac{n-1}{n} x^{n-1} + \frac{n-1}{2} x^{n-1} = \frac{d}{dx}[x^{n-1}b^f(x)].$$

Integrate both sides to get

$$b^f(x) = \left(\frac{n-1}{n^2} + \frac{n-1}{2n}\right) x,$$

which equals to the desired expression.

(C) Let $g(n) = (1 - 1/n)(1 + 2/n)$. Then

$$g'(n) = -1 + \frac{4}{n},$$

hence $g'(n) \leq 0$ for $n \geq 4$.

**Solution 10.2.4.3.**

(A) Let $(b_1(v), b_2(v))$ be a profile of bidding strategies. It is obvious that $b_i(v) \leq v$ in equilibrium. Suppose $b_1(1) < 1$. Then for player 2 with $v = 1$, his best response is $b_1(1) < b_2(1) < 1$. However, player 1 with $v = 1$ can profitably deviate to a bid $b$ such that $b_2(1) < b < 1$. Hence any profile with $b_i(1) \neq 1$ will not be a mutual best response.

(B) Suppose $b_1(1) = 1$ and $b_1(2) = x$. For player 2 with $v = 2$, his optimal bid is $b_2(2) = x + \epsilon < 2$ if $x < 2$ for some small enough $\epsilon$. But then player 1 with $v = 2$ will want to deviate to $x + \epsilon < x + \epsilon < 2$, since the deviation costs only arbitrarily little but earns $2 - (x + \epsilon')$ with probability 1. Suppose $b_1(2) = 2$. Then player 1 with $v = 2$ has a profitable deviation to any $x \in (1, 2)$, since he then has positive probability to earn some utility instead of zero for sure.(Recall that in equilibrium $b_2(1) = 1$.) Hence in equilibrium the bidder with $v = 2$ will not bid any deterministic value.

(C) Suppose the equilibrium bid of bidder 1 with $v = 2$ is a distribution $G(\cdot)$ with support $[1, x]$. Then for bidder 2 to bid $G(\cdot)$, it must be that bidder 2 is indifferent between any bidding in $[1, x]$. In particular, we have

$$0.5(2 - b) + 0.5G(b)(2 - b) = 0.5$$

where the right hand side is the expected utility for a bidder with $v = 2$ to bid 1. Rearrange we can get

$$G(b) = \frac{b - 1}{2 - b}.$$

The upper bound $x$ is defined through $G(x) = 1$, hence $x = 1.5$.

(D) The equilibrium payoff of a buyer with $v = 1$ is 0, which is the same as the second price auction in which they bid truthfully. The equilibrium payoff of a buyer with $v = 2$ is 0.5, since when the opponent uses $G(\cdot)$, every bid $b \in [1, 1.5]$ yields the same expected utility. The value is also the same to a second price auction's $0.5(2 - 1) + 0.25(2 - 2)$.

The expected revenue in the second price auction is $0.75 \times 1 + 0.25 \times 2 = 1.25$. Let $m_i^f(v)$ be the expected payment of a bidder in the first price auction. Then

$$m_i^f(v) = P(Win)v - EU(v).$$

So $m_i^f(1) = 0.25$ and $m_i^f(2) = 1$.

Then the expected revenue is

$$E_{v_1,v_2}[m_1^f(v_1) + m_1^f(v_2)] = 0.25 \times (0.25 + 0.25) + 0.5 \times (0.25 + 1) + 0.25 \times (1 + 1) = 1.25,$$

establishing revenue equivalence.

**Solution 10.2.4.4.**

(A) The argument that $b(1) = 1$ in equilibrium is exactly the same as before. For a bidder with $v = 2$, assume $G(\cdot)$ with support $(1, x]$ is the equilibrium. Then similar to before, we have

$$(1-p)(2-b) + pG(b)(2-b) = 1-p$$

where $1-p$ is the conditional probability that the opponent is a low type when the bidder is the high type. This rearranges to

$$G(b) = \frac{1-p}{p}\frac{b-1}{2-b}.$$

Using $G(x) = 1$ we can get $x = 1 + p$.

(B) For the second price auction, when the values are $(V_1, V_2) = (2, 2)$, the seller gets 2, otherwise the seller gets 1, hence the expected revenue is

$$(1 - \frac{p}{2})1 + \frac{p}{2}2 = 1 + 0.5p.$$

For the first price auction, let $m_i^f(v)$ be the equilibrium payment for a bidder with value $v$ with the equilibrium strategy derived in (A). Since $m_i^f(v) = P(\text{Win}|v)v - EU(v)$, where $P(\text{Win}|v)$ is the equilibrium probability of winning and $EU(v)$ the equilibrium payoff for type $v$. Hence $m_i^f(1) = 0.5p$ and $m_i^f(2) = (1 - p + 0.5p)2 - (1 + p) = 1$. The expected revenue for the firm is thus

$$E_{v_1,v_2}[m_1^f(v_1) + m_1^f(v_2)] = \frac{p}{2}(0.5p + 0.5p) + 2\frac{1-p}{2}(0.5p + 1) + \frac{p}{2}(2) = 1 + 0.5p$$

Hence the first price auction and the second price auction yield the same revenue for the seller.

**Solution 10.2.4.5.**

(A) The argument that $b(1) = 1$ is the same as exercise 3(A). To characterize the equilibrium bidding for $v = 2, 3$, we need to compute the posteriors first. If $v_i = 2$, then the conditional probability that the bidder $-i$'s value is also 2 is

$$P(v_{-i} = 2|v_i = 2) = P(\text{Urn } 1)P(\text{Draw } 2|\text{Urn } 1) + P(\text{Urn } 2)P(\text{Draw } 2|\text{Urn } 2) = 0.5.$$

Similarly,

$$P(v_{-i} = 1|v_i = 2) = P(v_{-i} = 3|v_i = 2) = 0.25$$
$$P(v_{-i} = 3|v_i = 3) = P(v_{-i} = 2|v_i = 3) = 0.5$$
$$P(v_{-i} = 1|v_i = 3) = 0.$$

Suppose in equilibrium $b(2) = x < 2$ is non-random. Then for a bidder with value 2, he can deviate to some $x + \epsilon < 2$, since his winning probability increases by 25% but the payment increased is arbitrarily small. Suppose $b(2) = 2$, then a bidder with $v = 2$ should underbid, so that he has 25% chance of winning a $v = 1$ bidder and gets a positive payoff. Similarly, $b(3)$ can not be deterministic either.

Now suppose the equilibrium bidding $b(2)$ is a distribution $G$ with support $(1, a]$. If he bids close to 1, he can get approximately $0.25(2 - 1)$. If he bids $a$, he wins unless the opponent is of $v = 3$, so he gets $0.75(2 - a)$. Since he plays a mixed strategy, he must be indifferent between all the pure strategies in the support of his mixed strategy. Hence

$$0.25 = 0.75(2 - a).$$

This solves to $a = 5/3$.

Next, assume the equilibrium strategy $b(3)$ is a distribution with support $(5/3, b]$. If he bids close to $5/3$, he wins unless the opponent is of $v = 3$, in which case he loses with probability one. The payoff is then $0.5(3 - 5/3)$. If he bids $b$, he wins with probability one, so the payoff is $1(3 - b)$. Indifference between any bid in $(5/3, b]$ implies

$$0.5(3 - 5/3) = (3 - b).$$

Hence $b = 7/3$.

(B) We provide a slightly different approach to compare revenues here(although the method in Exercise 4 still works.) The logic is to compute the equilibrium payoffs of the bidders. Since both auctions both allocate efficiently, the social surplus is equal. If the bidders in one auction have higher expected payoffs then the other auction, it will imply that the expected equilibrium payment to the seller is lower. Hence the revenue generated is lower.

For the second price auction, truthful bidding continues to be dominant. Under such equilibrium,

$$\begin{aligned}
&EU_i^S(1) = 0 \\
&EU_i^S(2) = P(v_{-i} = 1|v_i = 2)(2 - 1) + P(v_{-i} = 2|v_i = 2)(2 - 2) = \frac{1}{4} \\
&EU_i^S(3) = P(v_{-i} = 2|v_i = 3)(3 - 2) + P(v_{-i} = 3|v_i = 3)(3 - 3) = \frac{1}{2}.
\end{aligned}$$

For the first price auction, by the computations in (A) we know that

$$EU_i^F(1) = 0$$
$$EU_i^F(2) = 0.25$$
$$EU_i^F(3) = 0.5(3 - \frac{5}{3}) = \frac{2}{3}.$$

That the second price auction generates a higher revenue then follows from $EU_i^F(3) > EU_i^S(3)$.

**Solution 10.2.4.6.** Given the form of $f(V_2|V_1)$, we need a case by case discussion. Suppose $V_1 \leq 1$. Then

$$F(V_2|V_1) = \int_0^{V_2} \frac{x}{V_1} dx = \frac{V_2^2}{2V_1}.$$

Hence

$$\frac{F(V_2'|V_1)}{F(V_2|V_1)} = \frac{V_2'^2}{V_2^2} = \frac{F(V_2'|V_1')}{F(V_2|V_1')}.$$

Suppose $V_1 \geq 1$ and $V_1 - 1 \leq V_2 \leq 1$ Then

$$F(V_2|V_1) = \int_{V_1-1}^{V_2} \frac{x - (V_1 - 1)}{2 - V_1} dx = \frac{(V_2 - V_1 + 1)^2}{2(2 - V_1)}.$$

Suppose $V_1 \geq 1$ and $1 \leq V_2 \leq V_1$, then

$$F(V_2|V_1) = \int_{V_1-1}^{1} \frac{x - (V_1 - 1)}{2 - V_1} dx + \int_1^{V_2} 1 dx = \frac{2 - V_1}{2} + V_2.$$

Hence, when $V_1 > V_1' \geq 1 > V_2 > V_2'$,

$$\frac{F(V_2'|V_1)}{F(V_2|V_1)} = \frac{(V_2' - V_1 + 1)^2}{(V_2 - V_1 + 1)^2} \leq \frac{(V_2' - V_1' + 1)^2}{(V_2 - V_1' + 1)^2} = \frac{F(V_2'|V_1')}{F(V_2|V_1')}.$$

When $V_1 > V_1' > V_2 > V_2' > 1$, we have

$$\frac{F(V_2'|V_1)}{F(V_2|V_1)} = \frac{(2 - V_1)/2 + V_2'}{(2 - V_1)/2 + V_2} \leq \frac{(2 - V_1')/2 + V_2'}{(2 - V_1')/2 + V_2} = \frac{F(V_2'|V_1')}{F(V_2|V_1')}.$$

One can proceed in this manner to show all the remaining cases, that is, when $V_1 > 1 > V_1' > V_2 > V_2'$ and when $V_1 > V_1' > V_2 > 1 > V_2'$, conditional stochastic dominance is satisfied.

**Solution 10.2.4.7.**

(A) This is done simply by plugging in $a = 0, b = 1$ into the given formula.

(B) Let the joint distribution of $(C, V_1)$ be $F(c, v_1)$. Since $V_1 = X_1 + C$ and $F(c, v_1) = P(C \le c, V_1 \le v_1)$, we have

$$F(c, v_1) = \begin{cases} c(v_1 - c), & c \le v_1 < 1 + c \\ 0, & v_1 < c \\ c, & v_1 \ge 1 + c \end{cases}$$

Hence, the density $f(c, v) = \partial^2 F / \partial c \partial v$ is given by

$$f(c, v_1) = \begin{cases} 1, & c \le v_1 < 1 + c \\ 0, & \text{otherwise} \end{cases}$$

(C) We divide the condition $C \le V_1 < 1 + C$ into two cases, one is $V_1 \in [0, 1], C \in [0, V_1]$, the other is $V_1 \in (1, 2], C \in [V_1 - 1, 1]$. Then it follows from (A) and (B) that

$$f(C|V_1) = \begin{cases} \frac{1}{V_1} & \text{if } V_1 \in [0, 1], C \in [0, V_1] \\ \frac{1}{2 - V_1} & \text{if } V_1 \in (1, 2], C \in [V_1 - 1, 1] \\ 0 & \text{otherwise} \end{cases}$$

(D) Since $f(V_2|V_1) = f(V_2, V_1)/f(V_1)$, we compute $f(V_2, V_1)$ first.

Note that

$$\begin{aligned} F(V_2, V_1) &= \int_0^1 P(\tilde{X}_1 + C \le V_1, \tilde{X}_2 + C \le V_2 | C) f(C) dC \\ &= \int_0^1 P(\tilde{X}_1 \le V_1 - C) P(\tilde{X}_2 \le V_2 - C) dC. \end{aligned}$$

Since

$$P(\tilde{X}_i \le V_i - C) = \begin{cases} 1 & V_i - C \ge 1 \\ V_i - C & 0 \le V_i - C \le 1 \\ 0 & V_i - C \le 0, \end{cases}$$

when $V_1 \le 1$ and $1 \le V_2 \le 1 + V_1$ we have

$$F(V_2, V_1) = \int_{V_2 - 1}^{V_1} (V_1 - C)(V_2 - C) dC + \int_0^{V_2 - 1} V_1 - C dC,$$

hence

$$f(V_2, V_1) = 1 + V_1 - V_2,$$

and

$$f(V_2|V_1) = \frac{1 + V_1 - V_2}{V_1}.$$

When $V_1 \le 1$ and $V_1 \le V_2 \le 1$, we have

$$F(V_2, V_1) = \int_0^{V_1} (V_1 - C)(V_2 - C) dC,$$

hence $f(V_2, V_1) = V_1$,so

$$f(V_2|V_1) = 1.$$

When $V_1 \leq 1$ and $0 \leq V_2 \leq V_1$, we have

$$F(V_2, V_1) = \int_0^{V_2} (V_1 - C)(V_2 - C)dC,$$

hence $f(V_2, V_1) = V_2$, so

$$f(V_2|V_1) = \frac{V_2}{V_1}.$$

The cases where $V_1 \geq 1$ can be done in a similar manner.

## 10.3 Bargaining

**Solution 10.3.1.**

(A) In the second round, the buyer proposes. He must propose at least $\delta P*$, otherwise the seller will decline. The buyer will then propose $\delta P^*$. Thus, in the first round where the seller proposes, he can only propose at most $\delta(1 - \delta P^*)$. Finally, we check that the the buyer will not want the seller to decline in the second period. If the bargaining goes to the second round, the buyer gets $\delta(1 - \delta P^*)$ in the equilibrium path. If the buyer proposes something that the seller will decline, he can only get $\delta^2 V^* < \delta^2(1 - P^*) < \delta(\delta - \delta P^*) < \delta(1 - \delta P^*)$. Hence the subgame perfect equilibrium path is that in the first period the seller proposes $\delta - \delta^2 P^*$ and the buyer accepts any value not higher than $\delta - \delta^2 P^*$. This is independent of $V^*$.

(B) In the second round, the seller proposes. He can only propose at most $1 - \delta V^*$, otherwise the buyer will decline. Since $\delta(1 - \delta V^*) > \delta^2(1 - V^*) > \delta^2 P^*$, the seller would rather propose $1 - \delta V^*$ than something that will be declined. Now in the first round the buyer will then propose $\delta(1 - \delta V^*)$ and the seller will accept. Hence the SPNE is that in the first round the buyer proposes $\delta(1 - \delta V^*)$ and the seller accepts any price not lower than $\delta(1 - \delta V^*)$, which is also independent of $P^*$.

(C) Assume that the seller moves first. If there are $T$ periods where $T$ is even, then the SPNE will be independent of $V^*$, if $T$ is odd then the SPNE will be independent of $P^*$.

**Solution 10.3.2.**

(A) In the $T$-th period, the buyer will get zero. In the $T - 1$-th period, the seller can thus claim $V_{T-1}$ for himself and the buyer will accept. In the $T - 2$-th period, the buyer will then offer the seller

$V_{T-1}$, and leave himself with $V_{T-2} - V_{T-1}$. Continue in this manner, the seller in the first period will claim $V_0 - V_1 + V_2 ... + V_{T-1}$ for himself and the rest to the buyer, and the buyer will accept.

(B) If the seller moves last, then in the $T-1$-th period the buyer can claim $V_{T-1}$ for himself and the seller will accept. In $T-2$-th period the seller will thus give the buyer $V_{T-1}$ and leave himself $V_{T-2} - V_{T-1}$. In the $T-3$-th period the buyer will thus give the seller $V_{T-2} - V_{T-1}$ and leave himself with $V_{T-3} - V_{T-2} + V_{T-1}$, and so on. Hence in the first period the seller will claim $V_0 - V_1 + ... - V_{T-1}$ for himself and the rest goes to the buyer, and the buyer will accept.

(C) By the result of (A) and (B) and making $V_0, V_1, ..., V_{T-1}$ approximate to each other, we see that if the seller moves first and the buyer moves last, one of the subgame perfect equilibrium outcome is that the seller takes everything. And if the seller moves last, the seller gets nothing.

However, in the next to last period, the one being proposed will get nothing, so both accept and reject are optimal actions. A more robust SPNE is that in the next to last period the proposer offers the minimum positive amount of the share to the other player and it will be strictly optimal for that player to accept.

**Solution 10.3.3.**

(A) The last mover can take the entire pie and his opponent can do nothing better than accepting it. The situation is similar to Exercise 10.3.2 above.

(B) When $\delta \to 1$, from Exercise 10.3.2(C) we see that the last person who can propose a positive amount of surplus will get approximately everything.

## 10.4 Efficient allocation with private information

**Solution 10.4.1.**

(A) Since the item is a private good, only one person can get it. If $V_i > V_{-i}$, the efficient allocation rule will give player $i$ the item, hence he takes all the social surplus. If $V_i < V_{-i}$, then he get zero, all the social surplus($V_{-i}$) goes to player $-i$. Hence

$$M_i(V) = V_{-i} 1_{V_i < V_{-i}}.$$

And

$$P_i(V_{-i}) = S_i^*(0, V_{-i}) = V_{-i}$$

since the social surplus will be $V_{-i}$ if $V_i = 0$.

(B) The incentive payment is

$$M_1(V) - P_1(V_2) = -V_2$$

since $V_i > V_{-i}$ implies $M_1(V) = 0$.

(C) The incentive payment for bidder 2 is

$$M_2(V) - P_2(V_1) = V_1 - V_1 = 0,$$

he pays zero.