The Analytics of Information and Uncertainty
Answers to Exercises and Excursions

Chapter 1: Elements of Decision Under Uncertainty

1.2 The Probability Distribution

Solution 1.2.1. The estimate of the probability of rain, prior to receiving the news, is 2/3. Consistency of this prior estimate with posterior beliefs, after arrival of new information, requires that:

$$50\%(1.0) + 30\%(0) + 20\%(0.5) = \frac{2}{3}$$

But this equation is not true. His current beliefs about the chance of rain are not consistent with his beliefs about what the arriving information will reveal.

Solution 1.2.2.

(A) His best estimate is

$$\frac{1}{3}(1) + \frac{1}{3}(0.5) + \frac{1}{3}(0) = \frac{1}{2}.$$

(B) His best estimate obviously remains 1/2. But his *confidence* that the true p actually is 1/2 has now increased, indeed that confidence is as high as it can possibly be.

(C) Once again his best estimate, for the purposes of a bet on the next toss of the coin, is $p = 1/2$. But he can have no confidence at all in that estimate. Indeed, since he has learned that the coin is either two headed or two tailed, he is absolutely sure that the true p is either 1 or 0, both equally likely; the true p *cannot* be 1/2.

1.4 Expected Utility Rule

Solution 1.4.1. Since he is maximizing here over his actions, U may be given an "ordinal" interpretation. Thus he could equally well maximize:

$$\hat{U} \equiv \ln U = \pi_1 \ln(1 + c_1) + \pi_2 \ln(1 + c_2)$$

This is the equivalent of maximizing expected utility if the "cardinal" utility function has the form:

$$v(c) = \ln(1 + c).$$

So in this case the individual's choices would be consistent with those of a von Neumann - Morgenstern expected-utility maximizer.

Solution 1.4.2.

(A) A typical indifference curve is given by

$$u = \pi_1 c_1^{1/2} + \pi_2 c^{1/2}$$

The indifference curve bends toward the origin because it is convex:

$$c_2 = (u - \pi c_1^{1/2})^2$$

has a positive second derivative. The indifference curve touches the axis because for any u, one can take $c_1 = 0, c_2 = (u/\pi_2)^2$, then (c_1, c_2) will be on the indifference curve with utility u and at the same time on the y-axis. Similarly, choosing $c_1 = (u/\pi_1)^2, c_2 = 0$, we will get a point (c_1, c_2) on the indifference curve and on the x-axis.

(B) Suppose $v(\cdot)$ is strictly concave. Take any $(c_1, c_2), (c_1', c_2')$ such that

$$U = \pi_1 v(c_1) + \pi_2 v(c_2) = \pi v(c_1') + (1 - \pi) v(c_2'),$$

where $0 < \pi < 1$. We have

$$\begin{aligned}
&U(\lambda c_1 + (1-\lambda)c_1', \lambda c_2 + (1-\lambda)c_2') \\
&= \pi v(\lambda c_1 + (1-\lambda)c_1') + (1-\pi)v(\lambda c_2 + (1-\lambda)c_2') \\
&> \pi(\lambda v(c_1) + (1-\lambda)v(c_1')) + (1-\pi)(\lambda v(c_2) + (1-\lambda)v(c_2')) \\
&= \lambda(\pi v(c_1) + (1-\pi)v(c_2)) + (1-\lambda)(\pi v(c_1') + (1-\pi)v(c_2')) \\
&= U,
\end{aligned}$$

where the inequality follows from strict concavity of v. Hence the convex combination is strictly preferred to either one of the original bundles.

Solution 1.4.3.

(A) Let

$$\hat{l} = \left(c_1, c_2; \frac{\pi_1}{\pi_1+\pi_2}, \frac{\pi_2}{\pi_1+\pi_2}\right).$$

Then by the result for 2-outcome lottery

$$U(\hat{l}) = \frac{\pi_1}{\pi_1+\pi_2}v(c_1) + \frac{\pi_2}{\pi_1+\pi_2}v(c_2).$$

Let $\overline{v} = \frac{\pi_1}{\pi_1+\pi_2}v(c_1) + \frac{\pi_2}{\pi_1+\pi_2}v(c_2)$. Then, using the fact that $v(m) = 0$ and $v(M) = 1$,

$$\begin{aligned}
U(l^*(\overline{v})) &= v(M)(\frac{\pi_1}{\pi_1+\pi_2}v(c_1) + \frac{\pi_2}{\pi_1+\pi_2}v(c_2)) + v(m)(1 - \frac{\pi_1}{\pi_1+\pi_2}v(c_1) - \frac{\pi_2}{\pi_1+\pi_2}v(c_2)) \\
&= \frac{\pi_1}{\pi_1+\pi_2}v(c_1) + \frac{\pi_2}{\pi_1+\pi_2}v(c_2) \\
&= U(\hat{l}).
\end{aligned}$$

As the two lotteries have the same expected utility, $\hat{l} \sim l^*(\overline{v})$.

(B) By (A) and the Independence Axiom, for any $c_3 \in C$

$$(\hat{l}, c_3; 1-\pi_3, \pi_3) \sim (l^*(\overline{v}), c_3; 1-\pi_3, \pi_3). \tag{2}$$

By definition, $c_3 \sim l^*(v(c_3))$. Apply the Independence Axiom to this relation and the lottery $l^*(\overline{v})$, we get

$$(l^*(\overline{v}), c_3; 1-\pi_3, \pi_3) \sim (l^*(\overline{v}), l^*(v(c_3)); 1-\pi_3, \pi_3).$$

(C) [Graph omitted.]

(D) The first is a compound lottery which simplifies to $(c_1, c_2, c_3, \pi_1, \pi_2, \pi_3)$. The second is a compound lottery which simplifies to

$$(M, m, (1-\pi_3)\overline{v} + \pi_3 v(c_3), (1-\pi_3)(1-\overline{v}) + \pi_3(1-v(c_3))).$$

The claim then follows from $(1-\pi_3)\overline{v} = (\pi_1+\pi_2)\overline{v} = \pi_1 v(c_1) + \pi_2 v(c_2)$.

In sum, we have shown that for an arbitrary 3-outcome lottery $(c_1, c_2, c_3; \pi_1, \pi_2, \pi_3)$,

$$U(c_1, c_2, c_3; \pi_1, \pi_2, \pi_3) = \sum_{i=1}^{3} \pi_i v(c_i).$$

(E) As an induction hypothesis, suppose for any s-outcome lottery,

$$U(c_1, ..., c_s; \pi_1, ..., \pi_s) = \sum_{i=1}^{s} \pi_i v(c_i).$$

Given any $s+1$-outcome lottery $(c_1, ..., c_{s+1}; \pi_1, ..., \pi_{s+1})$, define

$$\hat{l} = (c_1, ..., c_s; \frac{\pi_1}{\sum_{i=1}^{s} \pi_i}, ..., \frac{\pi_s}{\sum_{i=1}^{s}} \pi_s).$$

Define

$$\overline{v} = \sum_{i=1}^{s} \frac{\pi_i}{\sum_{j=1}^{s} \pi_j} v(c_i).$$

Then the induction hypothesis will imply in a similar fashion

$$\hat{l} = l^*(\overline{v}).$$

Now we can apply the same arguments through (B) to (D) to show the Expected Utility Rule holds for any lottery with $s+1$-outcomes.

1.5 Risk Aversion

Solution 1.5.1.

(A) As (i), (ii), (iv), (vi) each have a negative second derivative, these utility functions are risk averse. Similarly, (iii) has a positive second derivative and this utility function is risk loving. Also, (v) has a zero second derivative, this utility function is risk neutral.

(B) For $c > a/2b$, the first order derivative $v'(c)$ becomes negative. Hence for sufficiently large wealth, the less the better, which may be unrealistic.

Solution 1.5.2.

(A) Agent 1 is risk neutral so he maximizes expected payoff, hence he chooses $G2$. For agent 2, we have

$$E[v_2(G1)] = 21.9$$
$$E[v_2(G2)] = 0.5(850)^{0.5} + 0.5(200)^{0.5} = 21.65$$
$$E[v_2(G3)] = 15.81.$$

Hence he chooses $G1$.

For agent 3, we have

$$E[v_3(G1)] = 230,400.$$
$$E[v_3(G2)] = 381,250.$$
$$E[v_3(G3)] = 500,000.$$

Hence he will choose $G3$.

(B) Agent 1 only maximizes expected payoff hence he will not diversify.

Let λ_i be the (positive) share allocated to G_i by agent i, $i = 2, 3$. A portfolio is then (λ_1, λ_2), such that $\lambda_1 + \lambda_2 \leq 1$. Agent i then solves

$$\max_{\lambda_1, \lambda_2} U_2(\lambda_1, \lambda_2) \equiv \max_{\lambda_1, \lambda_2} E[v_i(\lambda_1 G1 + \lambda_2 G2 + (1 - \lambda_1 - \lambda_2) G3)].$$

As $G2$, $G3$ are perfectly correlated, agent 2's expected utility from diversification is

$$U_2(\lambda_1, \lambda_2) = 0.5(\lambda_1 480 + \lambda_2 850 + (1 - \lambda_1 - \lambda_2)1000)^{1/2} + 0.5(\lambda_1 480 + \lambda_2 200)^{1/2}.$$

Note that

$$\left.\frac{\partial U_2}{\partial \lambda_1}\right|_{(1,0)} < 0 < \left.\frac{\partial U_2}{\partial \lambda_2}\right|_{(1,0)}.$$

When $(\lambda_1, \lambda_2) = (1, 0)$, agent 2 is in possession of one share of $G1$ and no $G2$ or $G3$, he can increase his expected utility by reducing his share of $G1$ and increasing in $G2$ (or $G3$). Hence, he will diversify.

Agent 3 is risk-loving, intuitively he will put all the eggs in a single basket. We can use exactly the same argument as above to show that he will not diversify. With

$$U_3(\lambda_1, \lambda_2) = 0.5\left(\lambda_1 480 + \lambda_2 850 + (1 - \lambda_1 - \lambda_2)1000\right)^2 + 0.5\left(\lambda_1 480 + \lambda_2 200\right)^2,$$

we see that

$$\left.\frac{\partial U_3}{\partial \lambda_1}\right|_{(0,0)} < 0, \qquad \left.\frac{\partial U_3}{\partial \lambda_2}\right|_{(0,0)} < 0,$$

which means that $\lambda_1 = \lambda_2 = 0$ is his optimal portfolio, i.e., he invests only in $G3$.

To conclude, the risk-averse agent is the only one that diversifies his portfolio.

Solution 1.5.3.

(A) On the left-hand side we depict the gamble $(120, 840; 5/6, 1/6)$ versus $c = 240$ for sure. On the right-hand side we depict the gamble $(0, 720; 5/6, 1/6)$ versus $c = 120$ for sure.

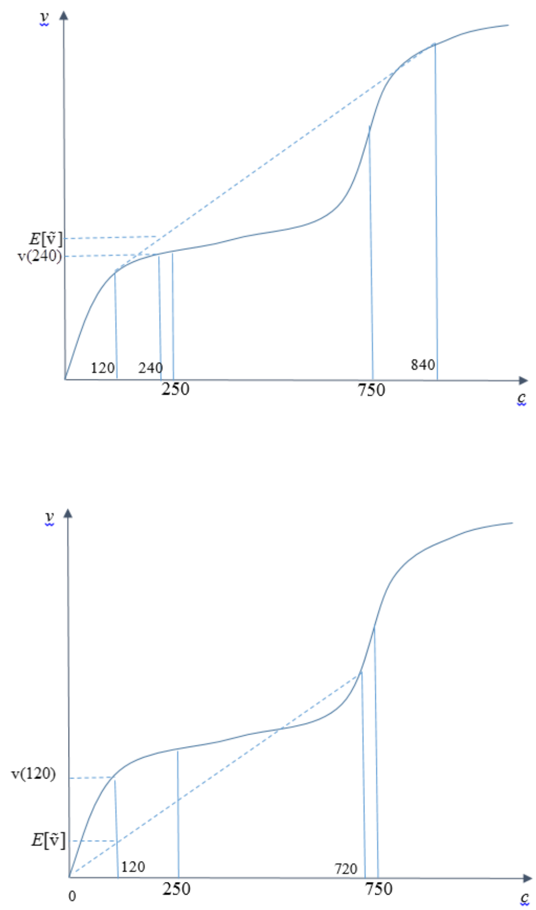
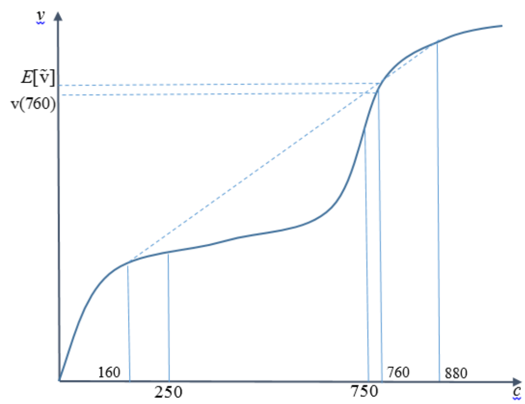


Figure 1: Ex 1.5.3(A)

(B) On the left-hand side we depict the gamble $(160, 880; 1/6, 5/6)$ versus $c = 760$ for sure. On the right-hand side we depict the gamble $(280, 1000; 1/6, 5/6)$ versus $c = 880$ for sure.



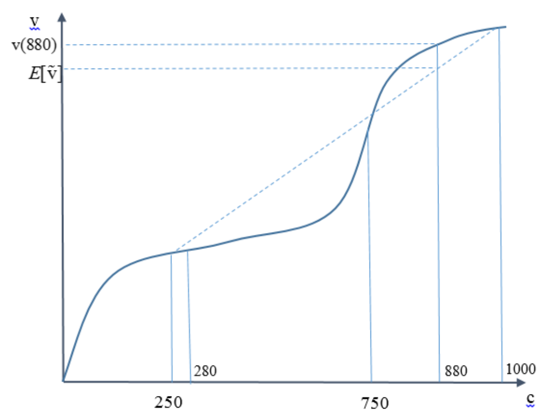


Figure 2: Ex 1.5.3(B)

(C) He can accept any lottery whose expected value is at least $v(500)$, which implies very large gains and losses in our picture.

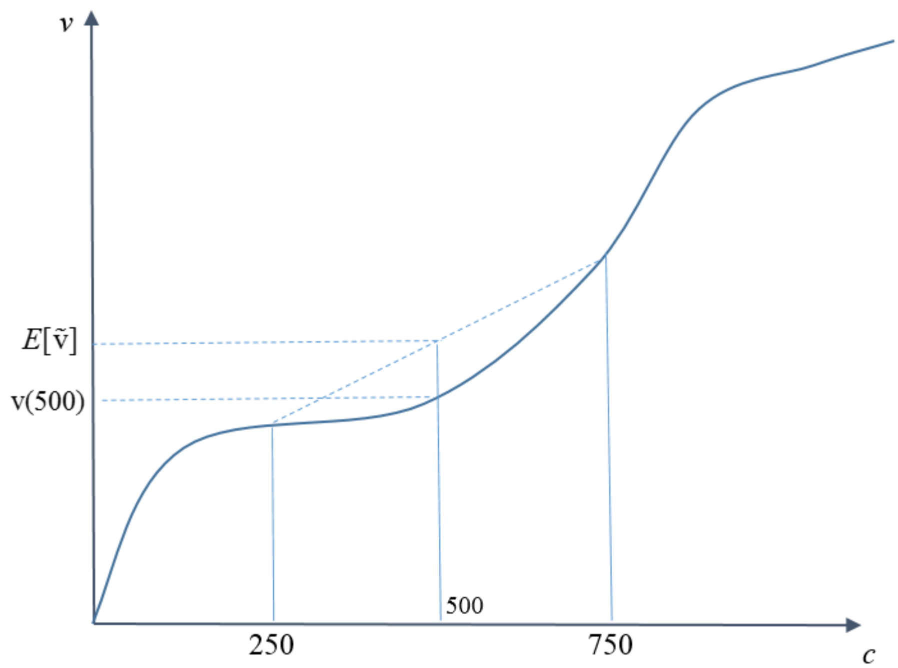


Figure 3: Ex 1.5.3(C)

Solution 1.5.4.

(A) Recall that $-v'/v'' = \alpha + \beta c$.

(i) With $\beta = 0$, rearrange to get

$$v''\alpha + v' = 0.$$

Hence for some constant k,

$$v'\alpha + v = M.$$

Writing $v' = dv/dc$ and rearrange again to obtain

$$\frac{dv}{M-v} = \frac{dc}{\alpha}.$$

Note that the left-hand side can be written as $-d\ln(k-v)$. Integrate both sides to obtain

$$\ln(M-v) = -\frac{c}{\alpha} + \ln N$$

for some constant N. Hence

$$M - v = Ne^{-c/\alpha}.$$

(ii) Rearrange to get

$$v''\beta c + v' = 0.$$

Multiply each side by c^k, where $(k+1)\beta = 1$, we then have

$$v''\beta c^{k+1} + v'c^k = 0.$$

The left-hand side can be written as $(v'\beta c^{k+1})'$, hence after integrating both sides we obtain

$$v'\beta c^{k+1} = N$$

for some constant N. Now we can rearrange and integrate again to solve for v, which will be the desired form.

(iii) Rearrange to get

$$\begin{aligned}
&v''c + v' = 0\\
&\Rightarrow (v'c)' = 0\\
&\Rightarrow v'c = N\\
&\Rightarrow v' = \frac{N}{c}\\
&\Rightarrow v = N\ln c + M.
\end{aligned}$$

(iv) Rearrange to get

$$v''(\alpha - c) + v' = 0. \tag{1}$$

Differentiate (1) with respect to c we get

$$v'''(\alpha - c) = 0$$

Hence $v(c)$ is a quadratic polynomial, which is of the form $v(c) = Nc^2 + Kc + D$. Now substitute it into (1) to get $2N\alpha + K = 0$, which is the desired equation.

(B) The utility function for case (i) is defined for all $c \in \mathbb{R}$, for cases (ii) and (iii) it is defined only for $c > 0$, and for case (iv) for $0 \leq c \leq \alpha$. Finally, $N > 0$ is equivalent to c being a desirable good (the more the better), and every well-behaved utility function should be increasing w.r.t. c.

Solution 1.5.5. Suppose you stake M dollars in a fair gamble where $M \leq 10000$.

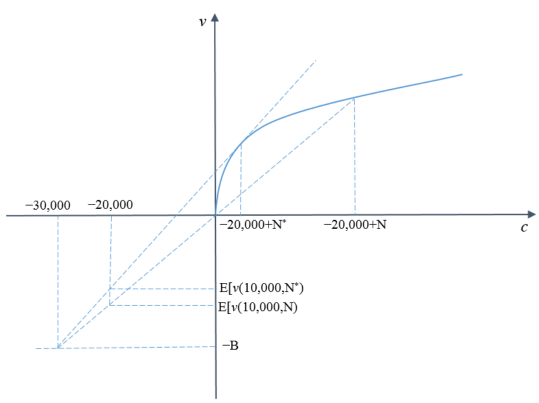


Figure 4: Ex 1.5.5

The gamble is of the form $(-M, N; p, 1-p)$, i.e., you lose M dollars with probability p, gain N dollars with probability $1-p$, and N, p are chosen such that

$$-Mp + N(1-p) = 0. \tag{2}$$

Then the prospect you are faced with will then be

$$(-20000 - M, -20000 + N, p, 1 - p).$$

Then the expected utility for choosing (M, N) will be

$$E[v(M, N)] = -Bp + v(-20000 + N)(1 - p)$$

Observe that for any fixed p the expected utility is increasing in N, and that by (2) N is increasing in M. Hence the agent will choose $M = 10000$. Secondly, if $N < 20000$ then the agent will get $-B$ for sure, hence he will choose $N \geq 20000$. The optimal N^* is depicted in Figure 1.5.5.

Solution 1.5.6. Let $I = [m_1, M_1] \times [m_2, M_2]$ be the set of consequences, $\Delta(I)$ be the space of lotteries over I. Let $\prec_I$ be a preference relation over I, $\prec$ be a preference relation over $\Delta(I)$ that is consistent with $\prec_I$, complete, continuous, and satisfies the independence axiom. Consider the lottery

$$l(\pi) = ((M_1, M_2), (m_1, m_2); \pi, 1 - \pi).$$

Now for each (a, b), define

$$v(a, b) = \pi$$

where

$$(a, b) \sim l(\pi).$$

The rest of the argument proceeds in a similar way to that in the text. First we need to show $v(a, b)$ is well-defined. That is, π exists and that $\pi_1 > \pi_2$ implies $l(\pi_1) \succ l(\pi_2)$, which can be shown by applying continuity and the independence axiom. The second step is to show $v(a, b)$ induces the same preference relation as $\prec_I$, which follows directly from the reference lottery technique.

Solution 1.5.7. First we solve the indirect utility function

$$U(I) = \max_{a,b} a^{1/2} b^{1/4}$$

s.t.

$$P_a a + P_b b = I.$$

From the Lagrangian we can obtain the first order conditions:

$$\frac{a^{-1/2} b^{1/4}}{2} = \lambda P_a$$
$$\frac{a^{1/2} b^{-3/4}}{4} = \lambda P_b$$
$$P_a a + P_b b = I.$$

We can then solve the optimal a, b as a function of (I, P_a, P_b). Plugging $a(I, P_a, P_b),\ b(I, P_a, P_b)$ into $v(a, b)$ to obtain

$$U(I, P_a, P_b) = \frac{(2/3)^{1/2}(1/3)^{1/4} I^{3/4}}{P_a^{1/2} P_b^{1/4}},$$

which is a strictly concave function of I.

(A) Since $U(I)$ is strictly concave in I,

$$U(50, 1, 1) > 0.5U(1, 1, 1) + 0.5U(99, 1, 1).$$

(B) The individual exhibits risk aversion to income risks because his utility function over income I is concave.

(C) Observe that

$$U(50, 64, 16) < 0.5U(50, 1, 16) + 0.5U(50, 81, 16).$$

Thus even though a 50:50 uncertainty of $P_a = 1$ or $P_a = 64$ has expected $P_a = 41$, he prefers to take the price risk over a certain $P_a = 50$.

The individual is risk averse toward income shocks but is risk loving toward price shocks, as his indirect utility function is convex in prices. A person can exhibit risk averse or risk loving attitude toward different parameters. Hence when we say a person is risk averse, we should be more precise as to indicate the parameter(s) the person is risk averse to.

Solution 1.5.8.

(A) Define $\overline{c} = E[\tilde{c}]$. By Taylor expansion, for any $c \neq \overline{c}$ there exists a c^* between c and $\overline{c}$ such that

$$v(c) = v(\overline{c}) + v'(\overline{c})(c - \overline{c}) + \frac{1}{2}v''(c^*)(c - \overline{c})^2.$$

If $v''(c) \leq 0$ for all c it follows that

$$v(c) \leq v(\overline{c}) + v'(\overline{c})(c - \overline{c}).$$

Thus

$$Ev(\tilde{c}) \leq v(\overline{c}) + v'(\overline{c})E(\tilde{c} - \overline{c}) = v(E(\tilde{c})).$$

(B) If $v''(c) < 0$ for all c it follows that

$$v(c) < v(\overline{c}) + v'(\overline{c})(c - \overline{c}) \quad \text{for } c \neq \overline{c}$$

Then as long as $\tilde{c} \neq \overline{c}$ with positive probability,

$$Ev(\tilde{c}) < v(E[\tilde{c}]).$$

Solution 1.5.9.

(A) Case $n = 2$ follows from definition. As an induction hypothesis suppose the inequality holds for $n = k$. For $n = k + 1$, we have

$$\begin{aligned} v\left(\sum_{i=1}^{k+1} \mu_i c_i\right) &= v\left((1-\mu_{k+1})(\sum_{i=1}^{k} \frac{\mu_i}{1-\mu_{k+1}} c_i) + \mu_{k+1} c_{k+1}\right) \\ &\geq (1-\mu_{k+1}) v\left(\sum_{i=1}^{k} \frac{\mu_i}{1-\mu_{k+1}} c_i\right) + \mu_{k+1} v(c_{k+1}) \\ &\geq (1-\mu_{k+1}) \sum_{i=1}^{k} \frac{\mu_i}{1-\mu_{k+1}} v(c_i) + \mu_{k+1} v(c_{k+1}) \\ &= \sum_{i=1}^{k+1} \mu_i v(c_i). \end{aligned}$$

(B) For any discrete random variable $\tilde{c}$, let $\mu_i = Pr(\tilde{c} = c_i)$. Then $E[\tilde{c}] = \sum \mu_i c_i$, $E[v(\tilde{c})] = \sum \mu_i v(c_i)$. Hence the inequality proven in (A) can be written as

$$v(E[\tilde{c}]) \geq E[v(\tilde{c})].$$

1.6 Utility Paradoxes and Rationality

Solution 1.6.1. A trickster could not profit if he had to offer them and pay off on gambles with positive returns, like those hypothetically presented in that example. However, he might be able to exploit their inconsistent choices if the funds backing the gambles come from some exogenous source, supposing that the trickster is in a position to direct who initially gets which. Specifically, let there be two individuals A and B (Alex and Bev) with identical endowments and preferences. The two exogenously supplied gambles, after stripping away the confusing framing of the question, amount to (i) \$250 certain, versus (ii) the prospect $(\$400, \$200; 0.25, 0.75)$. Suppose that Alex had indicated a preference for option (i) in the first version of the question and Bev a preference for (ii) in the second version. Then the trickster need only arrange matters so that Alex and Bev each initially receives his/her non-preferred option. Having done this he can say:

(To Alex): You had indicated a preference for 200+50 with certainty (call it option 1a) over \$200 plus a 25% chance of winning an extra \$200 (call it option 1b). You now have option 1b. I will take that off your hands and give you 1a instead, except that since you definitely prefer 1a you should be willing to sweeten the deal a little for me and give me just \$1.

(To Bev): You had indicated that receiving $\$400 - \$150$ with certainty (call it option 2a) is less desired than receiving \$400 subject to a \$75% chance of losing \$200 (call it option 2b). You now have option 2a. I will take that off your hands and give you 2b instead, except that since you definitely prefer 2b you should be willing to sweeten the deal a little for me and give me just \$1.

Given that 1a and 2a are identical, as are 1b and 2b, by "re-framing" the two gambles the trickster has been able to make a middleman's profit.

Remark 1. If Alex and Bev are both risk averse, each of them actually should prefer the certainty option (i) to the fair gamble (ii). Thus, in the described exchange Alex really gains; it is Bev who loses out, owing to the confusing framing of the question.

Solution 1.6.2.

(A) If you think there are more black balls than yellow balls, then you should choose black, otherwise you should choose red.

(B) If you think there are more black balls than yellow balls, then you should choose red, otherwise you should choose black.

(C) Suppose you choose black in (A). Formally, this means

$$(0, 100, 0; \frac{1}{3}, b, y) \succ (100, 0, 0; \frac{1}{3}, b, y),$$

where $b = Pr(\text{black}), y = Pr(\text{yellow})$. We can write them equivalently as compound lotteries

$$(1-y)(0,100;\frac{1}{3(1-y)},\frac{b}{1-y}) + y(0;1) \succ (1-y)(100,0;\frac{1}{3(1-y)},\frac{b}{1-y}) + y(0;1).$$

It then follows from the independence axiom that

$$(0,100;\frac{1}{3(1-y)},\frac{b}{1-y}) \succ (100,0;\frac{1}{3(1-y)},\frac{b}{1-y}).$$

By the independence axiom again, we have

$$(1-y)(0,100;\frac{1}{3(1-y)},\frac{b}{1-y})+y(100;1) \succ (1-y)(100,0;\frac{1}{3(1-y)},\frac{b}{1-y})+y(100;1).$$

This is equivalent to

$$(0,100,100;\frac{1}{3},b,y) \succ (100,0,100;\frac{1}{3},b,y),$$

or, the agent will choose red in (B).

Similarly, if the agent chooses red in (A), independence axiom will imply that he will choose black in (B).

(D) The Ellsberg paradox shows that individuals do not necessarily follow the independence axiom. There are a number of theories, such as ambiguity aversion, deceit aversion, that try to explain the Ellsberg paradox, and they will lead to different utility functions other than the von-Neumann Morganstein expected utility which is linear in probabilities. However, as mentioned in the text, behaviors inconsistent with the standard theory may be susceptible to money pump tricks, and this is often used as a justification to the standard theory.

Solution 1.6.3.

(A) Yes. Write the four lotteries (payoff is in terms of millions) considered in the text as follows:

$$A : (1, 1, 1; 0.1, 0.89, 0.01)$$
$$B : (5, 1, 0; 0.1, 0.89, 0.01)$$
$$C : (1, 0, 1; 0.1, 0.89, 0.01)$$
$$D : (5, 0, 0; 0.1, 0.89, 0.01).$$

The Allias paradox claims that $A \succ B$ and $D \succ C$. Consider the following two lotteries:

$$E : (0, 5; 1/11, 10/11)$$
$$F : (0, 0; 1/11, 10/11).$$

Then $A \succ B$ implies $A \succ E$ by the independence axiom, as $A = (A, A; 0.11, 0.89)$ and $B = (E, A; 0.11, 0.89)$. However, $A \succ E$ will imply $C \succ D$ by the independence axiom, as $C = (A, F; 0.11, 0.89)$ and $D = (E, F; 0.11, 0.89)$. Hence the Allais paradox violates the independence axiom.

(B) First you prepare a 100 faced die. As $A \succ B$, one can buy B from the agent with a price of $1 - \epsilon$ for a small enough ϵ. As $D \succ C$, one can buy C from the agent by offering him D. After the trade has been set, roll the die. Suppose the number is $1 \sim 10$, then from lottery B you gain 5 and from lottery C you gains 1, but the agent's lottery D makes you pay 5. Suppose the number is $11 \sim 99$, then from lottery B you gain 1. Suppose the number is 100, then from lottery C you also gain one. Hence, no matter what the outcome of the die is, you always get a net transfer of 1, but the only payment you make to the other agent is $1 - \epsilon$. Hence you can get ϵ for sure.

Solution 1.6.4.

(A) The indirect utility function is defined as

$$\hat{v}(p) = \max_{x,y} x + \alpha \ln y$$

s.t. $x + py = W$. Plugging in the budget constraint we get

$$\hat{v}(p) = \max_{y}(W + \alpha \ln y - py)$$

(B) Take FOC w.r.t. y to obtain

$$p = \frac{\alpha}{y}.$$

Hence

$$\hat{v}(p) = W - \alpha + a \ln \alpha - \alpha \ln p.$$

(C)

$$\frac{d^2\hat{v}}{dp^2} = \alpha \frac{1}{p^2} > 0.$$

(D) The situations leading respectively to the $v(x, y)$ and $\hat{v}(p)$ functions vary with regard to: (i) the source of uncertainty and, more importantly, (ii) the scope of allowable action once the uncertainty is resolved. In speaking of the $v(x, y)$ function as representing risk aversion with respect to good y, the source of uncertainty was possible variation in the person's endowment of good y alone, the other good x being held constant. And implicit in the definition of $v(x, y)$ is that the individual can take no further action once endowed with smaller or larger amounts of good y. With the $\hat{v}(p)$ function, the source of uncertainty for the individual is not his endowment (which is held fixed at W) but rather possible variations in price p. And, what is crucial, implicit in the $\hat{v}(p)$ function is that the individual is allowed now to respond to variations in p by optimally adjusting his x, y consumption pattern after the uncertainty has been resolved. The more extreme the price variation, the greater the gain from such ex post optimal adjustments. Hence, unless an individual is highly risk averse with respect to uncertainty about x and y, he is likely to prefer price variability.